Multi-Prize Contests as Incentive Mechanisms for the Provision of Public Goods with Heterogenous Agents

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Abstract

We analyze if and how multi-prize Tullock contests can be used to guarantee efficient contributions to a public good when agents are heterogenous both with respect to the costs of production of the public good and with respect to the utility from its consumption. With two types of individuals, efficiency can be guaranteed if the following conditions are met: (i) the contest designer can use at least two prizes different from zero, (ii) there is a sufficient number of individuals of each type or types are sufficiently similar and (iii) the reservation utility of the individuals resulting from non-participation is sufficiently low. For a large class of problems it turns out that the optimal prize structure is not monotonic.
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1 Introduction

Contests are mechanisms to promote incentives for the provision of non-rival goods that are frequently observed in practice. For example information goods like basic research are non-rival in consumption, and for political or contractual reasons the market mechanism is not applied in a substantial number of cases. In practice, incentives to invest in research are usually provided by means of mechanisms that have the structure of a contest: the chances to receive tenure positions, awards, or other forms of funding are determined by, for example, publications or other measures of scientific output, and it is the relative position of a single researcher compared to his peer group that determines his success. The value and number of the offered tenure positions or awards determines the resources or time spend on research by each individual. Typically individuals have different productivities or opportunity costs of research. These differences have important consequences for the incentive effects of a contest because efficiency requires that (potentially) each type of individual invests a different amount of time.

Most of the literature on contest design has focused on the analysis of single-prize contests. As has been mentioned in the literature, with only a single prize it is almost impossible to set efficient incentives if individuals are heterogenous (Lazear and Rosen (1981)). This observation is the starting point for this paper where we explore whether it is possible to shape incentives efficiently if the contest designer can use more than one prize and how the efficient prize structure looks if it exists. We use a model with two different types of individuals and two prizes in the main part of the paper and show how the results extend to the more general case of an arbitrary number of types with an arbitrary number of prizes in the appendix.

The methods applied in this paper are based on the theory of contests and rent-seeking pioneered for example by Tullock (1980). We analyze the prize structure for the special case of a nested Tullock function introduced by Clark and Riis (1998b) for the analysis of a standard rent-seeking game. This contest-success function has the advantage that the contest designer needs no individual-specific information in order to implement it. The results of this paper therefore hold for the case of asymmetric information where the contest designer knows the types and their fractions within the population, but not the identity of a specific individual.

One specific property of the Clark and Riis formulation is that the contestants invest only once and have the chance to win at most one of multiple prizes. This property fits nicely for our application of basic research when it comes to employment decisions because individuals can get at most one position at a university or a research institute at a time. It does not fit very well for the analysis of research grants where an application has to be written for every single grant, and a single individual can get more than one grant at a time.

Multi-prize contests can alternatively be modeled with a fully discriminating contest success function. This implies that the contestant with the ith highest effort will win the ith prize with certainty. These types of contests are discussed for example in Clark and Riis (1998a) and Moldovanu and Sela (2001) who focus on the optimality of multiple prizes if the objective is to maximize aggregate effort.

The idea to use relative performance measures to increase the incentives to invest in the voluntary provision of a public good has been described by Morgan (2000) who analyzes the provision of a public good by means of a lottery. Here the problem of underprovision of funds to finance an efficient quantity of the good is addressed but the production of the public good is not an issue. His main result is that in the limit for a large number of individuals a lottery contest can generate efficient incentives for voluntary funding.

One property of Morgan's model that makes it attractive is that budget-balance by the lottery organizer is 'internalized' in the problem because the prize is financed out of contributions, which, on the other hand explains the inefficiency (compared to the first best) of his mechanism for all finite populations. We address the budget-balance problem in a more standard way[1] by allowing the contest designer to charge a (lump-sum) fee to finance the prize sum. Under asymmetric information the maximum fee that can be used is determined by the minimum difference between the utility level received by participating in the contest and a reservation utility that can be enforced by the individual when he does not participate.[2] It is a well-known result from the literature that the exact specification of the reservation utility plays a crucial role. If the contest designer has sufficient coercive power to force the individuals into participation, budget balance will not be a problem. However, with voluntary participation budget balance can turn out to be a problem.

Because Morgan's focus is on fundraising for the production of public goods, not the process of production itself, he abstracts from the problem of heterogeneity of individuals in their abilities to provide the good and the associated sorting problems. The same is true for Clark and Riis (1998a) and Moldovanu and Sela (2001) who focus on aggregate effort. However, especially for the provision of basic research an equally important problem is the selection of the right types of individuals doing research. This latter problem is analyzed in Kolmar and Wagener (2004). They analyze the public good provision problem with individuals who differ in their productivities, whereby the production function is assumed to be linear. Furthermore the authors restrict the class to single-prize contests. They characterize whether a

[1] A remark to clarify the purpose of this paper can be made at this point. From a technical point of view, Morgan's as well as our contest are indirect mechanisms, and one knows from the direct-revelation principle that such a mechanism can at most be able to implement the allocations a direct mechanism can. Given that most real institutions do not have the character of a direct mechanism, studying indirect mechanisms that resemble real-world institutions however allows it to better understand the efficiency properties of mechanisms that can be found in reality.

[2] See Makowski and Mezzetti (1994) for the details of this approach.

‘tenure’ contest can provide efficient incentives and show that if an efficient mechanism exists the contest designer has a tradeoff between the size of the prize and the inventive intensity provided by the contest.
Abstracting from the specific problems imposed by public goods, Lazear and Rosen (1981) have shown that with heterogenous individuals it is impossible to efficiently sort individuals into ‘leagues’ for the case of two types of individuals and a single prize (or, more precisely, a single prize wedge). As our analysis will show, this result is not robust with respect to the number of prizes. As it is mentioned by Lazear and Rosen, abstracting from individual participation constraints their two-prize contest is structurally equivalent to a single-prize contest because only *differences* between prizes (‘prize spreads’) matter for incentive reasons.[3] The second prize in their paper can be interpreted as a minimum wage as it will be received for sure. A truly two-prize contest would require at least three contestants and a prize structure of the form (first, second, and others) (whereby others could be interpreted as a participation fee or bonus or just be set equal to zero). To be precise, only the difference between first (second) prize and participation fee/bonus sets the incentives to invest effort in the contests. We show that it is possible to efficiently solve the incentive problem for the case of two types, at least three individuals and two prizes. Given that both types are risk neutral, any spread in the prize structure that leaves its expected value unchanged leaves individual incentives unchanged. Hence, as long as the types differ in their marginal probabilities of winning the different prizes, incentives for the different types of individuals can be controlled separately if the number of prize spreads is sufficiently large. Moreover, for a large class of problems the optimal prize scheme has a rather surprising non-monotonic structure, requiring the second prize to be lower than the third prize.
The paper is organized as follows. In Section 2 we introduce the model where a public good can be provided by individuals with different convex cost functions. In Section 3 we characterize the set of efficient allocations. Section 4 demonstrates that voluntary provision will lead to a sub-optimal provision of the public good. Section 5 characterizes the conditions for the efficiency of a multi-prize contest, the concavity of the implied optimization problems, and the properties to balance the budget of the contest designer. Section 6 concludes.

2 The model

Take an economy with $N \geq 3$ individuals (‘researchers’), indexed by $i = 1, \ldots, N$. These individuals have different abilities for producing a non-rival or public (because we do not focus on mechanisms based on exclusion) good (‘scientific knowledge’),

[3]This is an application of the well-studied principle that the class of efficient mechanisms is uniquely determined up to a constant of integration. See, for example, Milgrom (2004).

whose quantity is denoted by y. x_i is individual i's contribution to the public good. $\mathbf{x} = \{x_1, ..., x_N\} = \{x_i, x_{-i}\}$ is the vector of contributions. For convenience we assume a linear mapping from individual contributions x_i into public-goods production y, $y = \sum_{l=1}^{N} x_l$, which implies that individual contributions $\mathbf{x}$ are perfect substitutes. A contribution of x_i involves the investment of a private good ('time') $t_i = c_i(x_i)$ for the individual. The mappings $c_i(.)$ from contributions to the private good ('cost functions') are assumed to be strictly convex, and we assume that $c_i(0) = c_i'(0) = 0$ and $\lim_{x_i \to \infty} c_i'(x_i) = \infty$. Each individual derives utility from the total amount of the public good, $v_i(y)$, and the consumption of a private good z_i ('leisure'). $v_i(y)$ is strictly concave in its argument with $v_i(0) = 0$. Effort as well as utility functions are twice continuously differentiable. The individuals have an endowment $\bar{z}$ of the private good which is assumed to be sufficiently large to guarantee an interior solution for all subsequent optimization problems.
We assume that there are two types of individuals, $N_H \geq 1$ H-type members and $N_L \geq 1$ L-type members, $N_H + N_L = N$. The sets of individuals of both types are denoted by $\mathcal{N}_H, \mathcal{N}_L$, $\mathcal{N} = \mathcal{N}_H \cup \mathcal{N}_L$. Individuals of the same type share the same utility as well as cost function, and we denote by x_H, x_L the contribution to the public good of a generic member of each group. An extension to N potentially different types can be found in Appendix C. Individuals decide individually and non-cooperatively about the amount of research x_i they undertake. They do so by maximizing their utility

$$u_i(x_i, z_i) = v_i\left(\sum_{j=1}^{N} x_j\right) + z_i \tag{1}$$

taking into account their budget constraint $\bar{z} = z_i + c_i(x_i)$. Inserting the budget constraint into the utility function yields

$$u_i(x_i) = v_i\left(\sum_{j=1}^{N} x_j\right) + \bar{z} - c_i(x_i). \tag{2}$$

The purpose of this paper is to design a contest that achieves efficiency even if the contest designer is asymmetrically informed about the type of an individual. To be more specific we assume that the contest designer knows the fractions of H- and L-type individuals in the population, $N_H/N, N_L/N$, but not the identity of a single individual. The individuals share this information and, in addition, know their own type. All this is common knowledge.

3 Efficiency

Assuming an interior solution with respect to the contribution to the public good, a first-best efficient allocation is characterized by

$$\max_{\mathbf{x}} \sum_{j=1}^{N} \left[v_j \left(\sum_{l=1}^{N} x_l \right) + \bar{z} - c_j \left(x_j \right) \right], \tag{3}$$

and the first-order conditions:

$$\sum_{j=1}^{N} \frac{\partial v_j \left(\sum_{l=1}^{N} x_l \right)}{\partial x_i} = \frac{\partial c_i \left(x_i \right)}{\partial x_i} \quad \forall i \in \mathcal{N}. \tag{4}$$

(4) is the standard Samuelson condition that determines the optimal contribution levels of all individuals i. From these conditions the efficient allocation $\mathbf{x}^*$ can be derived. Note that the FOCs imply that $c_i' \left(x_i^* \right) = c_j' \left(x_j^* \right)$ for all j, i. With different marginal-cost functions this implies that different individuals should choose different contributions. From the strict concavity of the utility and the strict convexity of the cost functions it follows that every member of the same type provides the same research effort to produce the public good, $x_i^* = x_j^* \quad \forall \quad i, j \in \mathcal{N}_K, K = H, L$. Without loss of generality assume that $x_H^* \geq x_L^*$.

4 Voluntary decentralized provision

A voluntary-contributions (Nash) equilibrium $\mathbf{x}^D$ is an allocation where individuals maximize utility by the choice of their contributions taking as given the contributions of all other individuals,

$$\max_{x_i} u_i \left(x_i, x_{-i} \right) = v_i \left(\sum_{l \neq i} x_l^D + x_i \right) + \bar{z} - c_i \left(x_i \right) \quad \forall i \in \mathcal{N}, \tag{5}$$

In a Nash equilibrium the following first-order conditions hold:

$$\frac{\partial v_i \left(\sum_{l=1}^{N} x_l^D \right)}{\partial x_i} = \frac{\partial c_i \left(x_i^D \right)}{\partial x_i} \quad \forall i \in \mathcal{N}. \tag{6}$$

Hence the individual incentives to provide the public good are misspecified compared to the efficient allocation because individuals do not internalize the positive externality imposed on the other individuals.

5 Implementation of the efficient allocation by means of contests

5.1 General set-up

Suppose that a contest is implemented such that the individual contributions determine the chance to win one of N prizes with value $w_j, j \in \mathcal{N}$ (measured in units of the private good), $\mathbf{w} = \{w_1, ..., w_N\} = \{w_i, w_{-i}\}$. As will become clear throughout the paper, with two types of individuals only two prizes are needed to induce efficient incentives, and only the prize differentials matter for incentive reasons. We can therefore without loss of generality normalize w_3 to w_N to equal zero. We need the general prize structure, however, for the case of more than two types of individuals analyzed in the appendix. The prizes are financed by means of lump-sum contributions t by the individuals, $\sum_i w_i = Nt$. These contributions can be either voluntary or imposed by a centralized authority (taxes). We will discuss the consequences of both interpretations throughout the text.

$P_i^j(\mathbf{x})$ is the probability that individual i wins prize j. Each can only win one prize, which implies that $\sum_{j=1}^N P_i^j = 1 = \sum_{i=1}^N P_i^j$.

The individual objective function becomes

$$\begin{aligned} u_i(x_i) &= v_i\left(\sum_{l=1}^N x_l\right) + \sum_{j=1}^N P_i^j(\mathbf{x})\, w_j + \bar{z} - t - c_i(x_i) \quad \forall i \in \mathcal{N} \\ &= v_i\left(\sum_{l=1}^N x_l\right) + \sum_{j=1}^2 P_i^j(\mathbf{x})\, w_j + \bar{z} - t - c_i(x_i) \quad \forall i \in \mathcal{N}. \end{aligned} \tag{7}$$

because of the convention $w_3 = w_4 = ... = w_N = 0$. To be more specific, we use a contest of the Tullock form as introduced by Clark and Riis (1998b) in order to induce individuals to implement the efficient allocation. The respective probabilities are

$$P_i^1(x_i, x_{-i}) = \frac{x_i}{\sum_{j=1}^N x_j}, \quad P_i^2(x_i, x_{-i}) = \sum_{j \neq i}\left(\frac{x_j}{\sum_{l=1}^N x_l}\frac{x_i}{\sum_{k \neq j} x_k}\right).$$

In this extension of the standard single-prize Tullock contest, the first prize is awarded as in a single-prize Tullock contest. The ratio of individual i's contributions and aggregate contributions determines his probability of winning the first prize. The probability of winning the second prize is constructed as follows. Individual $j \neq i$ wins the first prize with probability $x_j / \sum_{k=1}^N x_k$. In this case, it is excluded from the contest, and the probability for individual i of winning the second prize in this contingency is calculated as in a Tullock contest without individual j. The total probability for individual i of winning the second prize is the sum over all other individuals j of the probability that individual j wins the first prize times the probability that individual i wins the second prize given that individual j has been excluded.[4]

The objective functions is then

$$u_i(x_i) = \frac{x_i}{\sum_{j=1}^N x_j} w_1 + \sum_{j \neq i} \left(\frac{x_j}{\sum_{l=1}^N x_l} \frac{x_i}{\sum_{k \neq j} x_k} \right) w_2 + v_i \left(\sum_{l=1}^N x_l \right) + \bar{z} - t - c_i(x_i). \quad (8)$$

Denote by $dP_i^j(x_i, x_{-i}) = \partial P_i^j(\mathbf{x}) / \partial x_i$. Given (8), the first-order conditions of individual i are

$$\sum_{j=1}^{2} dP_i^j(x_i, x_{-i}) w_j + \frac{\partial v_i \left(x_i + \sum_{l \neq i} x_l \right)}{\partial x_i} - \frac{\partial c_i(x_i)}{\partial x_i} = 0 \quad \forall i \in \mathcal{N}. \quad (9)$$

A Nash-equilibrium is a vector $\mathbf{x}^n = \{x_1^n, ..., x_N^n\}$ such that for all i the FOCs hold at x_i^n, given x_{-i}^n.

Combining equations 4 and 9, a necessary condition for the efficiency of a Nash equilibrium is $\mathbf{x}^n = \mathbf{x}^*$, which implies that

$$\sum_{j=1}^{2} dP_i^j(x_i^*, x_{-i}^*) w_j = \sum_{k \neq i} \frac{\partial v_k \left(\sum_{l=1}^N x_l^* \right)}{\partial x_i} \quad \forall i \in \mathcal{N}, \quad (10)$$

which simply states that the marginal benefit from the contest should equal the marginal externality imposed by the individuals at the efficient allocation. The marginal externalities at the optimum will for convenience also be denoted by $EX_i(x_i^*, x_{-i}^*)$ in the following. Given that individuals of the same type behave identically in equilibrium, (10) can be simplified as follows:

$$\begin{aligned} dP_H^1(\mathbf{x}^*) w_1 + dP_H^2(\mathbf{x}^*) w_2 &= EX_H(\mathbf{x}^*), \qquad \forall i \in \mathcal{N}_H, \\ dP_L^1(\mathbf{x}^*) w_1 + dP_L^2(\mathbf{x}^*) w_2 &= EX_L(\mathbf{x}^*), \qquad \forall i \in \mathcal{N}_L. \end{aligned} \quad (11)$$

Because the first-order conditions are identical for all individuals of the same type, (11) effectively generates a system of 2 equations with 2 unknowns (the prizes). Written differently, the individuals' FOC form a system of linear equations of the general form

$$\begin{pmatrix} dP_H^1(\mathbf{x}^*) & dP_H^2(\mathbf{x}^*) \\ dP_L^1(\mathbf{x}^*) & dP_L^2(\mathbf{x}^*) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} EX_H(\mathbf{x}^*) \\ EX_L(\mathbf{x}^*) \end{pmatrix} \quad (12)$$

which for convenience is written as $\mathbf{Pw} = \mathbf{EX}$. Such a system has at least one solution $\mathbf{w}^*$ if and only if $\text{Rank}[\mathbf{P}] = \text{Rank}[\mathbf{P}|\mathbf{EX}]$. For the case of an extended Tullock function the following Lemma holds.

[4] For more details on this multiple-prize contest-success function see Clark and Riis (1996) and Clark and Riis (1998b).

Lemma: If $x_L^* \neq x_H^*$ and $N \geq 3$, Rank$[\mathbf{P}]$ = Rank$[\mathbf{P}|\mathbf{EX}] = 2$ for the extended Tullock function.

Proof: See Appendix A.
In order to guarantee the implementation of the efficient allocation, it is furthermore required that the prize structure $\mathbf{w}^*$ is such that the individuals' decision problems have a (global) maximum at the efficient allocation. A sufficient condition is that the decision problem be concave. We will check later if the optimal prize structure fulfills this requirement.
The special case of identical individuals follows immediately. If all individuals are identical, the rank of $\mathbf{P}$ as well as of $\mathbf{P}|\mathbf{EX}$ is one, which implies that efficient prizes can be found. In this special case, the concavity of the single-prize Tullock function is also guaranteed. Given that $x_H^* = x_L^*$ at the optimum, all entries into $\mathbf{P}$ are identical, and efficiency can be reached using a single-prize contest $w_1, w_2 = ... = w_N = 0$. However, every prize scheme $w_1 + k, w_2 = ... = w_N = k, k \geq 0$ induces the same incentives, which shows that the class of efficient prize structures is potentially much larger.[5]
On the other hand, with only one prize spread and two different types of individuals, Lazear and Rosen's (1981) finding that sorting is impossible with a Tullock CSF can be (generically) extended to the case of a public good.

Result 1: Assume a single-prize contest with two types of individuals and a Tullock CSF. Then it is generically impossible to implement an interior optimum with $x_H^* > 0, x_L^* > 0$.

Proof: In a symmetric equilibrium where each individual of the same type contributes an equal amount to the public good, the optimality conditions become

$$\frac{(N_H - 1)x_H^* + (N_L)x_L^*}{(N_H x_H^* + N_L x_L^*)^2} w_1 = EX_H \tag{13}$$

for the H-type, and

$$\frac{N_H x_H^* + (N_L - 1)x_L^*}{(N_H x_H^* + N_L x_L^*)^2} w_1 = EX_L \tag{14}$$

for the L-type. The H-type condition can only be fulfilled if

$$w_1 = \frac{(N_H x_H^* + N_L x_L^*)^2}{(N_H - 1)x_H^* + N_L x_L^*} EX_H.$$

Inserting this condition into the type-L condition yields, after some rearrangements,

$$\frac{x_H^*}{x_L^*} = \frac{\partial v_H(\mathbf{x}^*)/\partial x_H}{\partial v_L(\mathbf{x}^*)/\partial x_L},$$

[5] This observation may be useful for an extension of the analysis in the direction of participation constraints and potentially risk aversion because multiple strictly positive prizes may improve the allocation of risks from the point of view of the individuals.

which will not be fulfilled in general. *q.e.d.*

For the general case of two prize spreads, (13) and (14) implicitly define linear functions in the prize space,

$$w_1^H(w_2,0) = \frac{EX_H^* - dP_H^{2*}w_2}{dP_H^{1*}},$$
$$w_1^L(w_2,0) = \frac{EX_L^* - dP_L^{2*}w_2}{dP_L^{1*}}.$$

For each point on this function each individual has efficient incentives to provide the public good, given that all other individuals behave efficiently. It is therefore necessary to have $w_1^L(w_2,0) = w_1^H(w_2,0)$ for overall efficiency. This condition can always be fulfilled as long as both functions are no parallels, which would imply that

$$\frac{dP_H^{2*}}{dP_H^{1*}} = \frac{dP_L^{2*}}{dP_L^{1*}},$$

which, according to the Lemma, can never be fulfilled with a Tullock CSF as long as $x_H^* \neq x_L^*$. Hence, the efficiency conditions intersect. Then, the efficient prize structure is given by

$$w_1^* = \frac{EX_H^* dP_L^{2*} - EX_L^* dP_H^{2*}}{dP_H^{1*}dP_L^{2*} - dP_L^{1*}dP_H^{2*}}, \quad w_2^* = \frac{EX_L^* dP_H^{1*} - EX_H^* dP_L^{1*}}{dP_H^{1*}dP_L^{2*} - dP_L^{1*}dP_H^{2*}}. \tag{15}$$

We can summarize our findings as follows:

Result 2: Assume a two-prize contest with two types of individuals and a Tullock CSF. A necessary condition for the implementation of efficient incentives is given by prize structure (15).

Corollary: The optimal first prize w_1^* will always be positive. The optimal second prize w_2^* will always be negative if $v'_H(y^*) = v'_L(y^*)$ $(EX_H^* = EX_L^*)$. It can become positive if $v'_H(y^*) > v'_L(y^*)$ $(EX_H^* < EX_L^*)$.

Proof: See Appendix B.

The explanation for the ability of a multi-prize contest to shape incentives efficiently and the surprising possibility of a negative second prize is the following. Given that the rank of the marginal-probability matrix is two, both types of individuals have differently-sloped iso-incentive curves in the prize space. It can be shown that the iso-incentive curve of the low-productivity type will always be steeper in w_2-w_1-space. Since the slope is the negative ratio of marginal probabilities of winning the second versus the first prize this means that the low-productivity type is relatively more affected by a change in the second prize [6]. This property can be exploited

[6] The low productivity individual will typically be more motivated in absolute terms by a change in the first prize than the high-productivity type because of the concavity of the first prize Tullock CSF. This will always be compensated by a sufficiently higher marginal probability of winning the second prize of the low-productivity type.

to shape incentives. Assume that w_2 is equal to zero and we can offer each type a first prize w_1^H and w_1^L such that both types provide the efficient quantities (full-information scenario). Typically $w_1^H \neq w_1^L$. Assume that $\frac{EX_H^*}{dP_H^{1*}} > \frac{EX_L^*}{dP_L^{1*}}$, meaning that the marginal utility from the public good is sufficiently high for the low-productivity type compared to the high-productivity type, and hence $w_1^H > w_1^L$. Increasing w_1^L marginally will increase incentives to provide for the low-productivity type. Hence if $w_1 = w_1^H$ the low-productivity type will overprovide. To discourage him from overproviding we need to introduce a second prize and make use of the differences in the ratio of marginal probabilities. By increasing the first prize and introducing a negative second prize we can balance incentives in such a way that both types provide the efficient quantities. We lower the second prize because this will hurt low-productivity types more. If $\frac{EX_H^*}{dP_H^{1*}} < \frac{EX_L^*}{dP_L^{1*}} \Leftrightarrow w_1^H < w_1^L$ and we decrease w_1^L the low-productivity type underprovides. This is the case if the marginal benefit from the public good is sufficiently small for this type. In this case the first prize should be decreased and the second increased to balance incentives. Here we need to penalize the high-productivity type and hence the first prize is decreased. This finding is important because it shows that in contrast to the first intuition it may be optimal to use non-monotonic prize schedules when individuals are heterogeneous.The following example illustrates these two cases:

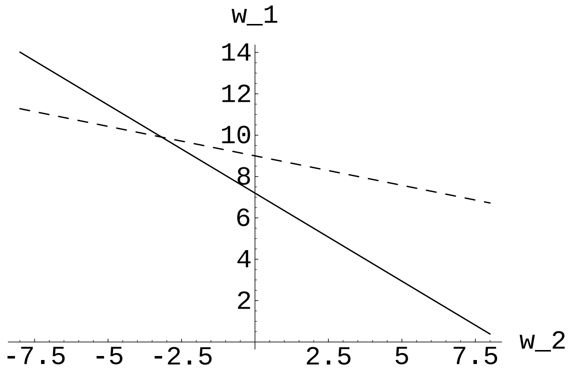

Figure 1: Indifference curves for H- and L- type and $EX_L^* = 1$

Figure 1 shows the efficient incentive-indifference curves of both types for two H- and two L-types with efficient provision levels $x_H^* = 2$ and $x_L^* = 1$ and identical externalities $EX_H^* = EX_L^* = 1$. This yields marginal probabilities of $dP_H^1 = 1/9, dP_L^1 = 5/36, dP_H^2 = 19/600, dP_L^2 = 71/600$, which leads to opportunity costs of w_1 in terms of w_2 of 57/200 (H) and 213/250 (L) respectively. The dashed line is the indifference curve for the H-type and the solid line is the indifference curve for the L-type as defined in (15). The efficient prize structure is then given at the intersection of both lines.

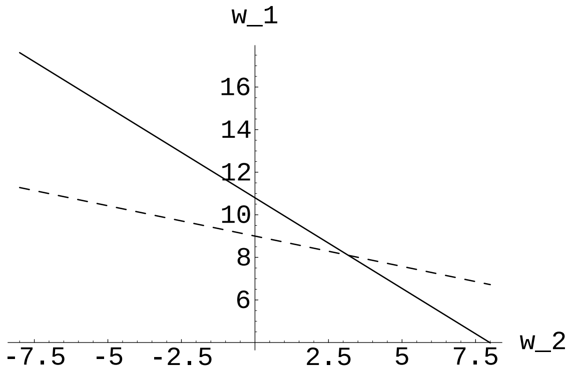

Figure 2: Indifference curves for H- and L- type and $EX_L^* = 1.5$

Figure 2 shows the efficient incentive-indifference curves for both types for the same specification as before except that EX_L increases to 1.5. In this scenario the low-productivity type has a relatively lower marginal benefit from the public good than the high-productivity type. The optimal second prize is positive in this case.

5.2 Concavity of the objective function

In this section we check whether the efficient prize structure (15) is compatible with the concavity of the individual maximization problems. A prize scheme leads to a concave individual optimization problem if

$$\underbrace{\sum_{j=1}^{2} \frac{\partial^2 P_i^j(x_i, x_{-i}^*)}{\partial x_i^2} w_j}_{\phi_i(x_i, x_{-i}^*, N_L, N_H)} \leq \left(\frac{\partial^2 c_i(x_i)}{\partial x_i^2} - \frac{\partial^2 v_i\left(x_i, x_{-i}^*\right)}{\partial x_i^2} \right) > 0 \; \forall i \in \mathcal{N}. \tag{16}$$

(16) defines a constraint on the prize scheme $\mathbf{w}^*$. Because we cannot define a finite and strictly positive lower bound for the right-hand side of (16) for general cost and utility functions, we characterize a sufficient condition for (16) to hold, namely that the left-hand side is negative or equal to zero.
If one denotes $\partial^2 P_i^j(x_i, x_{-i}^*)/\partial x_i^2$ by $ddP_i^j(x_i, x_{-i}^*)$, and evaluating at (15), the left-hand side of (16) becomes

$$\begin{aligned} \phi_i(x_i, \mathbf{x}^*, N_L, N_H) = \quad & \frac{ddP_i^1(x_i, x_{-i}^*)(EX_H^* dP_L^{2*} - EX_L^* dP_H^{2*})}{dP_H^{1*} dP_L^{2*} - dP_L^{1*} dP_H^{2*}} \\ + \quad & \frac{ddP_i^2(x_i, x_{-i}^*)(EX_L^* dP_H^{1*} - EX_H^* dP_L^{1*})}{dP_H^{1*} dP_L^{2*} - dP_L^{1*} dP_H^{2*}}, i = H, L. \end{aligned} \tag{17}$$

Unfortunately, (17) is not unambiguously negative for any combination of group-sizes and externality structures as the following simulation demonstrates. It is assumed that the marginal externalities are equal for both types of individuals and

that there is one H-type individual. [7]
Without loss of generality x_L^* has been normalized to be equal to 1. The following table displays the signs of $\phi_L(x_i, \mathbf{x}^*, N_L, N_H)$ if $EX_H^* = EX_L^*$ for different numbers of L-type individuals (rows) and different values for x_H^* (columns) if $N_H = 1$. The high-productivity type's second-order conditions are satisfied for all these parameter values.

$N_L \diagdown x_H^*$	1	2	3	4	5	6
2	-	+	+	+	+	+
4	-	-	+	+	+	+
6	-	-	-	-	+	+
8	-	-	-	-	-	+
10	-	-	-	-	-	-

As one can see, high values of x_H^* together with low values of N_L are likely to cause the wrong sign of the second-order conditions for the L-type. The reason is that there is no competitive pressure within the H-type group, which implies that incentives for the H-type individual can only be generated by means of L-type individuals. This causes a potential conflict regarding the L-type incentives that is becoming the more severe the larger the differential $x_H^* - x_L^*$ and the lower the number of L-type individuals is: large differences in contributions create large differences in the probabilities to win the prize, and low numbers of L-type individuals imply low competitive pressure within this group.
The case with only one H-type individual is a worst-case-scenario because it eliminates competition within this type-group. Increasing N_H to 2 leads to the following signs for $\phi_L(x_i, \mathbf{x}^*, N_L, N_H)$ where we have increased the maximum difference $x_H^* - x_L^*$ to 10

$N_L \diagdown x_H^*$	1	3	5	7	9	11
2	-	-	-	+	+	+
4	-	-	-	-	+	+
6	-	-	-	-	-	-
8	-	-	-	-	-	-
10	-	-	-	-	-	-

As one can see, the general pattern that large differentials in contributions together with small numbers of L-type individuals may cause a positive sign of $\phi_L(x_i, \mathbf{x}^*, N_L, N_H)$ remains unchanged. However, the number of cases where the second-order conditions

[7] Note that as we restrain from using functional specifications the effect of a change in N_L or N_H on $\mathbf{x}^*$ cannot be included in the simulation. Hence going downwards from one cell to another cannot be interpreted as a change in N_L only but in combination with a change in the productivities such as to keep x_H^* constant. A horizontal move can be interpreted as a change in productivity only though, with N_L held constant.

fail to hold is substantially reduced because of the stronger competition between H-type individuals.
The general intuition with respect to the sign of ϕ_i can be summarized as follows: the second-order conditions characterize a maximum if either $|x^*_H - x^*_L|$ is relatively small, the number of L-type and H-type individuals is relatively large, or both. In cases where either $|x^*_H - x^*_L|$ is sufficiently large or the number of H-type or L-type individuals is sufficiently small, the first-order conditions may characterize a minimum. To make this intuition more precise, Result 3 summarizes results for the case of symmetric externality structures whereas Result 4 summarizes results for the case of identical group sizes.

Result 3: Assume that $EX^*_H = EX^*_L$ (individuals differ only with respect to their cost functions). We then get the following limit behavior.

1. For all $N_L = aN^0_L$ and $N_H = aN^0_H$, $a > 0$, $\lim_{a\to\infty} \phi_H(x_i, \mathbf{x}^*, N_L, N_H) = \lim_{a\to\infty} \phi_L(x_i, \mathbf{x}^*, N_L, N_H) < 0$.
2. $\lim_{N_L\to\infty} \phi_H(x_i, \mathbf{x}^*, N_L, N_H) = \lim_{N_L\to\infty} \phi_L(x_i, \mathbf{x}^*, N_L, N_H) < 0$.
3. $\lim_{N_H\to\infty} \phi_H(x_i, \mathbf{x}^*, N_L, N_H) = \lim_{N_H\to\infty} \phi_L(x_i, \mathbf{x}^*, N_L, N_H) < 0$.
4. $\max[\lim_{x^*_H\to\infty} \phi_H(x_i, \mathbf{x}^*, N_L, N_H), \lim_{x^*_H\to\infty} \phi_L(x_i, \mathbf{x}^*, N_L, N_H)] < 0 \Leftrightarrow N_H \geq 3$.
5. $\phi_H(x_i, \mathbf{x}^*, N, N) < 0$, $\phi_L(x_i, \mathbf{x}^*, N, N) < 0$ for $N_H \geq 3$.

Results 3.1-3.3 characterize different limit behaviors with respect to population size. The picture portrayed by these results shows that efficiency can be achieved by means of a multi-prize contest if the number of individuals is sufficiently large. By the same token, Results 3.4 and 3.5 show that the concavity of the objective functions is also guaranteed if the optimal investment of the more efficient H-type is sufficiently large and the number of H-type individuals exceeds 2, or if there is an equal number of L- and H-type individuals.

Result 4: For general externality structures and $N_H = N_L = N$ we get the following limit behavior.

1. $\lim_{N\to\infty} \phi_H(x_i, \mathbf{x}^*, N_L, N_H) = \lim_{N\to\infty} \phi_L(x_i, \mathbf{x}^*, N_L, N_H) < 0 \Leftrightarrow \frac{v'_L(y^*)}{v'_H(y^*)} < \frac{x^*_H}{x^*_L}$.
2. $\max[\lim_{x^*_H\to\infty} \phi_H(x_i, \mathbf{x}^*, N_L, N_H), \lim_{x^*_H\to\infty} \phi_L(x_i, \mathbf{x}^*, N_L, N_H)] < 0 \Leftrightarrow \frac{v'_H(y^*)}{v'_L(y^*)} \geq \frac{2N-1}{(N-1)N}$.

The message contained in Result 4 is that the direct marginal benefit from an increase in public good $v'_i(y)$ should not be too large for the low-productivity individual compared to the high-productivity individual. Otherwise the low-productivity individual cannot be restrained from overproviding the public good. For example, if

$x_H^* = 2x_L^*$, the latter's direct marginal benefit should not exceed twice that of the high productivity type for $N \to \infty$. On the other hand, for $\lim_{x_H^* \to \infty}$, given that there are two individuals of each type, the ratio of direct marginal benefits of the high- compared to the low-productivity type should not fall short of 3/2. As N increases, this restriction is becoming less severe.

5.3 Budget balance

It follows from the direct-revelation principle that a multi-prize contest can at most be as efficient as a direct mechanism. We have seen in the prior sections that for a large class of problems it is in fact possible to shape incentives efficiently by means of contests. However, it remains to be shown that the efficient prize structure balances the budget of the contest designer and is compatible with the participation constraints of the individuals.

Budget balance requires that $\sum w_i^* = Nt$. We analyze whether the participation constraints can be fulfilled in all situations where they can be fulfilled for an efficient direct mechanism. If this is the case we have found an indirect mechanism that is incentive-equivalent to the optimal direct mechanism.

We follow the literature and distinguish between three types of participation constraints, compulsory, voluntary *ex-ante*, and voluntary *interim*.

1. We first analyze whether budget balance is possible with compulsory participation in the contest. We interpret the compulsory power of the contest designer as an arbitrarily low value of the individual reservation utility, which makes the implementation of the efficient contest almost trivial: the minimum contribution (tax) required to balance the budget is $t^{cp} = (\sum w_i^*)/N$.[8] The implementation of a mechanism that leads to Pareto efficiency however, need not be a Pareto-improvement for all individuals.

2. Second we analyze the *ex-ante* case where individuals can commit to participate in the contest before they learn their types, and participation is voluntary at this

[8]See also Arrow (1979) for the case of continuous Gradstein (1994) for the case of a discrete public goods who show possibility results for the implementation of efficient mechanisms if the reservation utility of the individuals is becoming arbitrarily small.

stage. The expected utility of an individual in this situation is equal to

$$\begin{aligned} E[u_i(\mathbf{x}^*)] &= \frac{N_H}{N}\left(v_H(\mathbf{x}^*) + \sum_j p_H^j(\mathbf{x}^*)w_j^* + \bar{z} - t - c_H(x_H^*)\right) \\ &\quad + \frac{N_L}{N}\left(v_L(\mathbf{x}^*) + \sum_j p_L^j(\mathbf{x}^*)w_j^* + \bar{z} - t - c_L(x_L^*)\right). \end{aligned} \tag{18}$$

We assume that the *ex-ante* reservation utility of the individuals is determined by the voluntary-contributions equilibrium. This specification can be justified if one assumes that the mechanism is only implemented by the unanimous support of the individuals, and that the economy returns to the model of voluntary contributions as soon as one individual votes against the mechanism. This utility is given by $E[u_i(\mathbf{x}^D)]$, where the expectations are formed as in (18). The participation constraint is therefore fulfilled if

$$E[u_i(\mathbf{x}^*)] - E[u_i(\mathbf{x}^D)] \geq 0.$$

Since $\mathbf{x}^*$ maximizes overall utility it also maximizes expected utility. Hence, and not surprisingly, the *ex-ante* participation constraints can always be fulfilled. For the class of problems for which this constraint is appropriate we can therefore deduce that a multiple-prize contest can be as efficient as a direct mechanism.

3. Third we analyze the *interim* case where individuals can commit to participate in the contest after they have learned their types but before the outcome of the contest is realized, and participation is voluntary at this stage. Their expected utilities are equal to

$$E_i[u_i(\mathbf{x}^*)] = v_i(\mathbf{x}^*) + \sum_j p_i^j(\mathbf{x}^*)w_j^* + \bar{z} - t - c_i(x_i^*), \quad i = H, L. \tag{19}$$

We again determine the reservation utility with recourse to the voluntary-contributions equilibrium and the associated utility levels $u_i(\mathbf{x}^D)$,

$$v_i(\mathbf{x}^*) + \sum_j p_i^j(\mathbf{x}^*)w_j^* + \bar{z} - t - c_i(x_i^*) \geq u_i(\mathbf{x}^D), \quad i = H, L.$$

Because of the information structure of this model the Bayesian direct expected-externality mechanism can always fulfill participation constraints.[9] The incentive neutral transfer component of such a mechanism which can only depend on signals of all other contestants is able to extract all surplus from the contestant, because the true signals of the other contestants contain the information necessary to deduce the type of the contestant.

[9]See Makowski, Mezzetti (1994).

The contest mechanism with lump-sum tax financing of the prize money is not necessarily on a par with the direct mechanism. Assume that $c_L(x_L) = ax_L^2$, $a > 1$, $c_H(x_H) = x_H^2$ and $v_H(\mathbf{x}) = v_L(\mathbf{x}) = \sqrt{\sum_{i=1}^{N} x_i}$. The participation constraint is always fulfilled for the high-productivity type for these parameters. The difference $E_i[u_i(\mathbf{x}^*)] - u_i(\mathbf{x}^D)$, $i = H, L$, will have the following sign for the low-productivity type, given $N_H = 2$:

$N_L \diagdown a$	2	12	22	32	42	52
2	+	-	-	-	-	-
4	+	-	-	-	-	-
6	+	+	-	-	-	-
8	+	+	+	-	-	-
10	+	+	+	+	+	-
12	+	+	+	+	+	+
14	+	+	+	+	+	+

Increasing N_H increases the number of cases where the participation constraint is fulfilled. Notice that as in the analysis of the second-order conditions, low numbers of competitors are problematic, as well as a large productivity difference. Hence if there are sufficiently many individuals of each type and the productivity difference is not too large, the contest achieves efficiency.

6 Conclusions

In this paper we have analyzed whether it is possible to provide efficient incentives for the decentralized production of a public good by means of a multi-prize contest if individuals differ with respect to their costs of production as well as with respect to their utility from the consumption of the public good. The specific advantage of a multiple-prize contest is that this mechanism does not rely on individual-specific information about productivities and preferences.

The general characterization of the problem has revealed that from an efficiency point of view only prize spreads matter. This finding generalizes the analysis of Lazear and Rosen (1981), who found a similar structure for the case of a two-prize contest. Generally speaking, the dimension of the space of endogenous variables has to be equal to the dimension of the space of control variables in order to guarantee efficiency. If the relevant control variables are *prize spreads*, this implies that the absolute number of control variables has to *exceed* the number of endogenous variables. In fact we are able to show that for the case of a *two-prize contest and at least three individuals* it is possible to implement the efficient allocation in a setup similar to Lazear and Rosen's.

In addition to the general efficiency result, the two central insights of the paper are as follows. (a) Optimal multi-prize contests may involve non-monotonic prize schemes. We have demonstrated for the case of two different cost types that the optimal prize scheme will in fact always be v-shaped if both types have the same marginal utility function for the public good. The reason for this surprising property is that the second prize primarily serves as a mechanism to prevent the low-productivity individuals from overinvestment. (b) Contrary to other incentive mechanisms, contests can only induce efficient incentives if there exists sufficient competition for the prizes. Hence, if there is a single high-productivity individual, it may be impossible to achieve efficiency because the only means by which this individual can be motivated is to induce the low-productivity individuals to invest more. Hence, if the cost-differential between types is becoming sufficiently large, this property may be in conflict with the concavity of the individuals' optimization problems. This finding points to an interesting implication of the analysis: if the within-group competition for the most-qualified individuals is too weak, it may be optimal to exempt them from a mechanism based on relative performance and to instead motivate them by conventional forms of incentive contracts.

Appendix

Part A

Proof that Rank[$\mathbf{P}$] = 2 for the case of two types of individuals and a population of at least 3. The following is the reduced $\mathbf{P}$-Matrix with only one row for each type and all prizes except 1 and 2 set equal to zero. (We showed that with two types the maximal rank is two and hence we can choose $N-2$ prizes and delete all identical rows.)

$$\begin{pmatrix} \frac{\partial P_H^1}{\partial x_H} & \frac{\partial P_H^2}{\partial x_H} \\ \frac{\partial P_L^1}{\partial x_L} & \frac{\partial P_L^2}{\partial x_L} \end{pmatrix} \tag{A.1}$$

Column Rank: First we show that the column rank is always equal to two if $x_H \neq x_L$: For this we show conditions under which there exists a constant a with the following property

$$\begin{aligned} \frac{\partial P_H^1}{\partial x_H} a &= \frac{\partial P_H^2}{\partial x_H} \\ \frac{\partial P_L^1}{\partial x_L} a &= \frac{\partial P_L^2}{\partial x_L} \end{aligned} \tag{A.2}$$

and that they are excluded by our assumptions.

The *first* condition is

$$x_H^* = x_L^* \qquad \wedge \qquad -x_L^* + N_H x_L^* + N_L x_L^* \neq 0. \tag{A.3}$$

This would be the case of only one type of individual in the population and of course the rank here is one. This case is excluded by the assumption that $x_L^* \neq x_H^*$.

The *second* condition is

$$x_H^* = 0 \quad \wedge \quad N_L = \frac{1}{2} \quad \wedge \quad x_L^* \neq 0. \tag{A.4}$$

Since N_L is at least one this condition is not relevant for our analysis and can be neglected.
The *third* condition is

$$x_H^* \neq 0 \quad \wedge \quad N_H = -\frac{N_L x_L^*}{x_H^*} \quad \wedge \quad x_L^* \neq 0. \tag{A.5}$$

This condition is not valid in our case as none of the above variables can be negative and hence the second part of the condition can never be fulfilled.
Fourth the rank will not be full if

$$x_H^* \neq 0 \quad \wedge \quad N_H = \frac{x_H^* + x_L^* - 2N_L x_L^*}{2x_H^*} \quad \wedge \quad x_H^* - x_L^* \neq 0. \tag{A.6}$$

Since N_L enters the second part of this condition with a negative sign, assuming $N_L = 1$ will be the least restrictive case. But with $N_L = 1$ the condition becomes $N_H = \frac{x_H^* - x_L^*}{2x_H^*} = \frac{1}{2} - \frac{x_L^*}{2x_H^*}$. This can never be greater or equal to one as is required for our N_H and hence the condition is not relevant.
Finally, the following condition has to be eliminated

$$\begin{aligned} & x_H^* \neq 0 \quad \wedge \\ & N_H = \frac{1}{2x_H^{*2}} \left(x_H^{*2} + 2x_H^* x_L^* (1 - N_L) - \sqrt{x_H^{*4} - 4x_H^{*3} x_L^* (1 - N_L) + 4x_H^{*2} x_L^{*2} (1 - N_L)} \right) \\ & \text{or } N_H = \frac{1}{2x_H^{*2}} \left(x_H^{*2} + 2x_H^* x_L^* (1 - N_L) + \sqrt{x_H^{*4} - 4x_H^{*3} x_L^* (1 - N_L) + 4x_H^{*2} x_L^{*2} (1 - N_L)} \right) \\ & \wedge \quad -x_L^* + N_H x_L^* + N_L x_L^* \neq 0 \end{aligned} \tag{A.7}$$

The second and third part here are little more tricky but can be eliminated by comparing the first derivatives with respect to N_L. First note that $N_L = 1$ will imply $N_H = 1$ which is ruled out by the assumption that we have at least three individuals. If now N_L is increased there will be an effect from the term $-2N_L x_H^* x_L^*$ and an effect from $\sqrt{x_H^{*4} - 4x_H^{*3} x_L^* (1 - N_L) + 4x_H^{*2} x_L^{*2} (1 - N_L)}$. The first will linearly decrease N_H and is hence not problematic. The second though is potentially positive so we have to see which effect dominates. Note that the second term is a concave function of N_L. Hence it is sufficient to compare slopes at $N_L = 1$. If the slope of the first term is greater in absolute value there will be no problem as there sum will be negative for all possible cases. The slope of the first term is

$$\frac{\partial G}{\partial N_L}|_{N_L=1} = -2x_H^* x_L^*. \tag{A.8}$$

The slope of the second term is equal to

$$\frac{\partial H}{\partial N_L}|_{N_L=1} = 2 (x_H^* - x_L^*) x_L^* = 2x_H^* x_L^* - 2x_L^{*2}. \tag{A.9}$$

Hence the negative term will always dominate and N_H is restricted to be less or equal to one which we rule out by assumption. Hence we have shown that the

column rank of the **P**-Matrix is always two with two types of individuals and at least three individuals.
Since the row rank is always equal to the column rank we have shown that the rank of **P** is equal to two.

q.e.d.

Part B

The optimal second prize given that marginal externalities are equal is

$$w_2^* = -\frac{(N_H + N_L - 1)(N_H x_H^* + (N_L - 1)x_L^*)^2((N_H - 1)x_H^* + N_L x_L^*)^2}{((2N_H - 1)x_H^* + (2N_L - 1)x_L^*)\left((N_H - 1)N_H x_H^{*2} + 2(N_H - 1)(N_L - 1)x_L^* x_H^* + (N_L - 1)N_L x_L^{*2}\right)} \tag{B.1}$$

which is clearly negative for $N_L, N_H \geq 2$.

The optimal second prize for all possible externality structures:

$$\begin{aligned} w_2^* = &- \frac{(N_H + N_L - 1)(N_H x_H^* + (N_L - 1)x_L^*)^2((N_H - 1)x_H^* + N_L x_L^*)^2}{(x_H^* - x_L^*)((2N_H - 1)x_H^* + (2N_L - 1)x_L^*)} * \\ &\frac{(v_L' x_H^* - v_H' x_L^*)}{\left((N_H - 1)N_H x_H^{*2} + 2(N_H - 1)(N_L - 1)x_L^* x_H^* + (N_L - 1)N_L x_L^{*2}\right)}. \end{aligned} \tag{B.2}$$

The sign here depends on v_L' and v_H' and hence the marginal benefit from investing in the public good for each type. If we assume that $x_H^* > x_L^*$ the second prize will become positive if

$$\frac{v_L'}{v_H'} < \frac{x_L^*}{x_H^*}. \tag{B.3}$$

Hence if the marginal benefit from the public good is sufficiently small for the low-productivity type compared to the high-productivity type a positive second prize is optimal.

Part C

In this appendix we give a general characterization of the problem to implement the optimal allocation $\mathbf{x}^*$ by means of a multi-prize contest. Assume that there are potentially N different types and N different prizes. Then, it is a straightforward extension of the model to show that a necessary condition for a multi-prize contest to implement the optimal allocation is the following:

$$\begin{pmatrix} dP_1^1(x_1^*, x_{-1}^*) & \cdots & dP_i^N(x_1^*, x_{-1}^*) \\ \vdots & \ddots & \vdots \\ dP_N^1(x_N^*, x_{-N}^*) & \cdots & dP_N^N(x_N^*, x_{-N}^*) \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix} = \begin{pmatrix} EX_1(x_1^*, x_{-1}^*) \\ \vdots \\ EX_N(x_N^*, x_{-N}^*) \end{pmatrix} \tag{C.1}$$

which as before is written as $\mathbf{Pw} = \mathbf{EX}$. Given that the optimization problem can be written in terms of prize spreads, it should not come as a surprise that it can be shown that the maximum rank of matrix $\mathbf{P}$ is $N-1$: Because $\sum_{j=1}^{N} P_i^j = 1 = \sum_{i=1}^{N} P_i^j$ for all $\mathbf{x}$, we know that

$$\sum_{j=1}^{N} dP_i^j = 0 \quad \Leftrightarrow \quad \sum_{j \neq N} dP_i^j = -dP_i^N, \tag{C.2}$$

which implies that the N-th column of $\mathbf{P}$ is a linear combination of the first $N-1$ columns. Hence, the maximum rank is $(N-1)$.

This finding implies that with N different individuals in the sense of different optimal x_i the contest mechanism cannot implement the efficient solution except if it happens that the augmented $\mathbf{P}|\mathbf{EX}$-matrix also has at most rank $N-1$.

This implies that if there are $k < N$ groups of individuals that differ with respect to their cost functions, the maximum rank of $\mathbf{P}$ is k. The rank of $\mathbf{P}|\mathbf{EX}$ depends on the externality structure imposed by the individuals.

An important consequence is that it will be impossible for the contest mechanism to implement the efficient allocation if there is a subgroup of individuals which are identical in their cost functions (up to a constant of integration) but differ in their utility for the public good $v_i(x_i)$. For illustration let them be called j and k and $v_j'(y) \neq v_k'(y)$ (by more than a constant of integration). Then the row j and k of matrix $\mathbf{P}$ are identical as both contestants should choose the same level of research $x_k^* = x_j^*$ by condition 4. Hence their marginal probabilities of success will be identical. In contrast the jth and kth scalar in vector $\mathbf{v}$, the marginal externality on all others, will be different. Accordingly there is no vector of prizes $\mathbf{w}$ which can implement the efficient allocation. If individuals have identical cost functions their utility functions for the public good must be identical, too, up to a constant of integration. This is not too surprising though, as individuals with the same costs but different utilities can be seen as different types as well.

Part D

Proof of Result 3: All parts of the result focus on the sign of $\phi_i(.)$ and therefore constitute sufficient conditions for the concavity of the objective functions. Using the Tullock function, for $EX_H^* = EX_L^*$ (meaning $v_H' = v_L'$), $\phi_i(.)$ reduces to:

$$\phi_H = -\frac{2v_H'(N_H + N_L - 1)}{(N_H\, x_H^* + (N_L - 1)\, x_L^*)((N_H - 1)\, x_H^* + n\, x_L^*)\,(N_H\, x_H^* + N_L\, x_L^*)\,((2\,N_H - 1)\, x_H^* + (2\,N_L - 1)\, x_L^*)}\,\Gamma \tag{C.1}$$

where:

$$\begin{aligned}
\Gamma \quad = \quad & \frac{(N_H - 1)\, N_H^2\,(1 + (N_H - 1)\, N_H)\, {x_H^*}^5}{((N_H - 1)\, N_H\, {x_H^*}^2 + 2\,(N_H - 1)\,(N_L - 1)\, x_H^*\, x_L^* + (N_L - 1)\, N_L\, {x_L^*}^2)} \\
+ \quad & \frac{(N_H - 1)\,(N_L + N_H\,(-3 + N_H\,(4 - 3\,N_L + N_H\,(5\,N_L - 4))))\, {x_H^*}^4\, X_L^*}{((N_H - 1)\, N_H\, {x_H^*}^2 + 2\,(N_H - 1)\,(N_L - 1)\, x_H^*\, x_L^* + (N_L - 1)\, N_L\, {x_L^*}^2)} \\
+ \quad & \frac{(N_H - 1)\,(2 - 3\,N_L - 2\,N_L^2 + N_H\,(-3 + (11 - 2\,N_L)\,N_L) + 2\,N_H^2\,(2 + N_L\,(-8 + 5\,N_L)))\, {x_H^*}^3\, {x_L^*}^2}{((N_H - 1)\, N_H\, {x_H^*}^2 + 2\,(N_H - 1)\,(N_L - 1)\, x_H^*\, x_L^* + (N_L - 1)\, N_L\, {x_L^*}^2)} \\
+ \quad & \frac{(-(N_L\,(-5 + N_L\,(7 + N_L))) + N_H\,(1 + N_L\,(-17 + 30\,N_L - 8\,N_L^2)) + N_H^2\,(-1 + 2\,N_L\,(6 + N_L\,(-12 + 5\,N_L))))\, {x_H^*}^2\, {x_L^*}^3}{((N_H - 1)\, N_H\, {x_H^*}^2 + 2\,(N_H - 1)\,(N_L - 1)\, x_H^*\, x_L^* + (N_L - 1)\, N_L\, {x_L^*}^2)} \\
+ \quad & \frac{N_L\,(2 + N_L\,(-10 + (11 - 2\,N_L)\,N_L) + N_H\,(-2 + (-2 + N_L)\,N_L\,(-6 + 5\,N_L)))\, x_H^*\, {X_L^*}^4 + (N_L - 1)\, N_L^2\,(1 + (-3 + N_L)\,N_L)\, {x_L^*}^5)}{((N_H - 1)\, N_H\, {x_H^*}^2 + 2\,(N_H - 1)\,(N_L - 1)\, x_H^*\, x_L^* + (N_L - 1)\, N_L\, {x_L^*}^2)}
\end{aligned}$$

and

$$\phi_L = -\frac{2\,v'_H\,(N_H + N_L - 1)}{(N_H\,x^*_H + (N_L - 1)\,x^*_N)\,((N_H - 1)\,x^*_H + N_L\,x^*_N)\,(N_H\,x^*_H + N_L\,x^*_N)\,((2\,N_H - 1)\,x^*_H + (2\,N_L - 1)\,x^*_N)}\,\Upsilon \tag{C.2}$$

where

$$\begin{aligned}
\Upsilon \;=\; & \frac{(N_H - 1)\,N_H^2\,(1 + (-3 + N_H)\,N_H)\,{x^*_H}^5}{((N_H - 1)\,N_H\,{x^*_H}^2 + 2\,(N_H - 1)\,(N_L - 1)\,x^*_H\,x^*_N + (N_L - 1)\,N_L\,{x^*_N}^2)} \\
+ \; & \frac{N_H\,(2 - 2\,N_L + N_H\,(-10 + 11\,N_H - 2\,N_H^2 + (-2 + N_H)\,(-6 + 5\,N_H)\,N_L))\,{x^*_H}^4\,x^*_N}{((N_H - 1)\,N_H\,{x^*_H}^2 + 2\,(N_H - 1)\,(N_L - 1)\,x^*_H\,x^*_N + (N_L - 1)\,N_L\,{x^*_N}^2)} \\
+ \; & \frac{(N_L - N_L^2 + N_H\,(N_L - 1)\,(-5 + 12\,N_L) + N_H^2\,(-7 + 6\,(5 - 4\,N_L)\,N_L) + N_H^3\,(-1 + 2\,N_L\,(-4 + 5\,N_L)))\,{x^*_H}^3\,{x^*_N}^2}{((N_H - 1)\,N_H\,{x^*_H}^2 + 2\,(N_H - 1)\,(N_L - 1)\,x^*_H\,x^*_N + (N_L - 1)\,N_L\,{x^*_N}^2)} \\
+ \; & \frac{((N_L - 1)\,(2 + N_L\,(-3 + 4\,N_L) + N_H\,(-3 + (11 - 16\,N_L)\,N_L) + 2\,N_H^2\,(-1 + N_L\,(-1 + 5\,N_L)))\,{x^*_H}^2\,{x^*_N}^3}{((N_H - 1)\,N_H\,{x^*_H}^2 + 2\,(N_H - 1)\,(N_L - 1)\,x^*_H\,x^*_N + (N_L - 1)\,N_L\,{x^*_N}^2)} \\
+ \; & \frac{(N_L - 1)\,(N_H - 3\,N_H\,N_L^2 + (-4 + 5\,N_H)\,N_L^3 + N_L\,(-3 + 4\,N_L))\,x^*_H\,{x^*_N}^4 + (N_L - 1)\,N_L^2\,(1 + (N_L - 1)\,N_L)\,{x^*_N}^5}{((N_H - 1)\,N_H\,{x^*_H}^2 + 2\,(N_H - 1)\,(N_L - 1)\,x^*_H\,x^*_N + (N_L - 1)\,N_L\,{x^*_N}^2)}
\end{aligned}$$

The denominators are unambiguously positive. The numerators can be divided by $-2v'_H(N_H + N_L - 1)$, which is unambiguously negative.

$$\begin{aligned}
\varphi_H \;=\; & (N_H - 1)\,N_H^2\,(1 + (N_H - 1)\,N_H)\,{x^*_H}^5 + (N_H - 1)\,(N_L + N_H\,(-3 + N_H\,(4 - 3\,N_L + N_H\,(-4 + 5\,N_L))))\,{x^*_H}^4\,x^*_L \\
+ \; & (N_H - 1)\,(2 - 3\,N_L - 2\,N_L^2 + N_H\,(-3 + (11 - 2\,N_L)\,N_L) + 2\,N_H^2\,(2 + N_L\,(-8 + 5\,N_L)))\,{x^*_H}^3\,{x^*_L}^2 \\
+ \; & (-(N_L\,(-5 + N_L\,(7 + N_L))) + N_H\,(1 + N_L\,(-17 + 30\,N_L - 8\,N_L^2)) + N_H^2\,(-1 + 2\,N_L\,(6 + N_L\,(-12 + 5\,N_L))))\,{x^*_H}^2\,{x^*_L}^3 \\
+ \; & N_L\,(2 + N_L\,(-10 + (11 - 2\,N_L)\,N_L) + +N_H\,(-2 + (-2 + N_L)\,N_L\,(-6 + 5\,N_L)))\,x^*_H\,{x^*_L}^4 \\
+ \; & (N_L - 1)\,N_L^2\,(1 + (-3 + N_L)\,N_L)\,{x^*_L}^5
\end{aligned}$$

$$\begin{aligned}
\varphi_L \;=\; & (N_H - 1)\,N_H^2\,(1 + (-3 + N_H)\,N_H)\,{x^*_H}^5 + N_H\,(2 - 2\,N_L + N_H\,(-10 + 11\,N_H - 2\,N_H^2 + (-2 + N_H)\,(-6 + 5\,N_H)\,N_L))\,{x^*_H}^4\,x^*_L \\
+ \; & (N_L - N_L^2 + N_H\,(N_L - 1)\,(-5 + 12\,N_L) + N_H^2\,(-7 + 6\,(5 - 4\,N_L)\,N_L) + N_H^3\,(-1 + 2\,N_L\,(-4 + 5\,N_L)))\,{x^*_H}^3\,{x^*_L}^2 \\
+ \; & (N_L - 1)\,(2 + N_L\,(-3 + 4\,N_L) + N_H\,(-3 + (11 - 16\,N_L)\,N_L) + 2\,N_H^2\,(-1 + N_L\,(-1 + 5\,N_L)))\,{x^*_H}^2\,{x^*_L}^3 \\
+ \; & (N_L - 1)\,(N_H - 3\,N_H\,N_L^2 + (-4 + 5\,N_H)\,N_L^3 + N_L\,(-3 + 4\,N_L))\,x^*_H\,{x^*_L}^4 + (N_L - 1)\,N_L^2\,(1 + (N_L - 1)\,N_L)\,{x^*_L}^5
\end{aligned}$$

If this remaining part is larger or equal to zero, the optimal prize scheme constitutes an individual maximum.

Result 3.1: Taking the limit of φ_H, φ_L for the replica economy, one arrives at $\text{sign}[\lim_{a\to\infty} \varphi_H] = \text{sign}[\lim_{a\to\infty} \varphi_L] = \text{sign}[N_H^0 x_H^* + N_L^0 x_L^*] > 0$.

Result 3.2: Taking the limit of φ_H, φ_L with respect to $N_L \to \infty$, one arrives at $\text{sign}[\lim_{N_L\to\infty} \varphi_H] = \text{sign}[\lim_{N_L\to\infty} \varphi_L] = \text{sign}[x_L^*] > 0$.

Result 3.3: Taking the limit of φ_H, φ_L with respect to $N_H \to \infty$, one arrives at $\text{sign}[\lim_{N_H\to\infty} \varphi_H] = \text{sign}[\lim_{N_H\to\infty} \varphi_L] = \text{sign}[x_H^*] > 0$.

Result 3.4: Taking the limit of φ_H, φ_L with respect to $x_H^* \to \infty$, one arrives at $\text{sign}[\lim_{x_H^*\to\infty} \varphi_H] = \text{sign}[(N_H)^3 - 2(N_H)^2 + 2N_H - 1]$, $\text{sign}[\lim_{x_H^*\to\infty} \varphi_L] = \text{sign}[(N_H)^3 - 4(N_H)^2 + 4N_H - 1]$. Both conditions only intersect at $N_H = 0$. The binding condition is fulfilled if $N_H \geq 3$.

Result 3.5: Inserting $N_H = N_L = N$ into φ_H, φ_L and divide by $(N - 1) > 0$ yields

$$\begin{aligned}
\widetilde{\varphi_H} = \;& 2x_H^{*3} x_L^{*2} - 2N x_H^* x_L^* (x_H^* + x_L^*)^3 + N^4 (x_H^* + x_L^*)^5 - N^3 (x_H^* + x_L^*)^4 (x_H^* + 3x_L^*) \\
+ \;& N^2 (x_H^* + x_L^*)^2 (x_H^{*3} + 2x_H^{*2} x_L^* + 8x_H^* x_L^{*2} + x_L^{*3}) \\
\widetilde{\varphi_L} = \;& 2x_H^{*2} x_L^{*3} - 2N x_H^* x_L^* (x_H^* + x_L^*)^3 + N^4 (x_H^* + x_L^*)^5 - N^3 (x_H^* + x_L^*)^4 (3x_H^* + x_L^*) \\
+ \;& N^2 (x_H^* + x_L^*)^2 (x_H^{*3} + 8x_H^{*2} x_L^* + 2x_H^* x_L^{*2} + x_L^{*3})
\end{aligned}$$

Simulations show that these conditions are unambiguously positive for $N \geq 3$.

q.e.d.

Proof of Result 4: The proof is similar to the one of Result 3. The denominator of $\phi_i(.)$ is unambiguously positive also for general externality structures:

$$\frac{(x_H^* - x_L^*)\,(N_H\,x_H^* + (N_L - 1)\,x_L^*)\,((N_H - 1)\,x_H^* + N_L\,x_L^*)\,(N_H\,x_H^* + N_L\,x_L^*)}{((-1 + 2\,N_H)\,x_H^* + (-1 + 2\,N_L)\,x_L^*)\,((N_H - 1)\,N_H\,{x_H^*}^2 + 2\,(N_H - 1)\,(N_L - 1)\,x_H^*\,x_L^* + (N_L - 1)\,N_L\,{x_L^*}^2)}$$

Hence, a sufficient condition for the concavity of the objective function is that the numerator of $\phi_i(.)$ is negative or zero. Call them φ_H, φ_L respectively.

Result 4.1: Taking the limit of φ_H, φ_L for $N \to \infty$, one arrives at $\text{sign}[\lim_{N\to\infty} \varphi_H] = \text{sign}[\lim_{N\to\infty} \varphi_L] = \text{sign}[v_L' x_L^* - v_H' x_H^*] < 0 \Leftrightarrow v_L'/v_H' < x_H^*/x_L^*$.

Result 4.2: Taking the limit of φ_H, φ_L for $x_H^* \to \infty$, one arrives at $\text{sign}[\lim_{x_H^*\to\infty} \varphi_H] = -\text{sign}[(N-1)^2 v_H' + N v_L']$, $\text{sign}[\lim_{x_H^*\to\infty} \varphi_L] = \text{sign}[(1-2N)v_L' + N(N-1)v_H']$. These conditions simplify to

$$\frac{v_H'}{v_L'} > \max\left[-\frac{N}{(N-1)^2}, \frac{2N-1}{(N-1)N}\right] = \frac{2N-1}{(N-1)N}.$$

q.e.d.

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