# The Choice of Prices vs. Quantities under Uncertainty*

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**Abstract**

This paper analyzes a model in which duopolistic firms first choose their strategy variable, either a price or a quantity, and compete afterwards while facing uncertain demand conditions. Contrary to the existing literature we show that due to the uncertainty, firms do not always choose a quantity, which is the variable that induces a smaller degree of competition. The reason is that demand uncertainty and the degree of competition have countervailing effects on variable choice. High uncertainty favors prices, while close substitutability favors quantities. Moreover, for intermediate values, a hybrid equilibrium exists.

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# 1 Introduction

The two classic papers in the theory of strategic interaction among firms are those by Cournot (1838) and Bertrand (1883). The first one proposes quantities as the strategy variable while the latter one suggests prices. It is well known since that time that quantities are the strategy variable that results in a lower degree of competition and therefore in higher profits of firms. So if firms are free to choose their strategy variable they would prefer quantities rather than prices. This result was first confirmed by Singh and Vives (1984) in a deterministic two-stage game in which firms first choose their strategy variable and compete afterwards. They show that quantities are a dominant action for both firms.[1]

However, a deterministic model might not be fully appropriate in this set-up because firms often face uncertainty at the time the strategy variable has to be chosen. For example, firms may be uncertain about the size of the market or about the distribution of consumers' reservation prices. Our analysis incorporates this aspect by introducing uncertainty via shocks that affect the slope and the intercept of the demand curve. In this set-up we show that the dominance of quantities does no longer hold. By contrast, a higher amount of uncertainty lowers firms' profits under quantity competition and favors prices. Moreover, we find that for an intermediate amount of uncertainty a "hybrid" equilibrium exists in which one firm selects a price and the other one a quantity.

We set out by looking only at one shock that enters the slope of the demand curve. We first demonstrate that these shocks affect the inverse demand curves of each firm in a non-linear way. The price decrease following a bad shock is larger than the price increase following a good shock of the same size. Thus, expected profits under quantity competition decrease in the amount of uncertainty. The same is not true for price competition since the direct demand curve is linearly affected by the shock and so expected profits under price competition stay constant. We show that the subgame perfect equilibrium under uncertainty is always unique. Since quantities have a competitive advantage relative to prices as they induce a smaller degree of competition, the degree of substitutability and

[1]For a similar but more general analysis, see Cheng (1985).

the degree of uncertainty have countervailing effects. Firms select prices as their strategic variable if uncertainty is high relative to the degree of competition and select quantities if the reverse holds true. We also prove that for every degree of competition a hybrid equilibrium in which one firm sets a price and the other one chooses a quantity emerges if both effects balance each other.[2]

We also analyze the case in which additionally to a shock entering the slope there is a shock affecting the intercept and the two shocks might be correlated. Here the same line of reasoning as before applies but rather than the variance here it is the covariance that drives the choice of firms' strategy variables.[3] If the covariance is positive and high relative to the degree of substitution both firms select prices rather than quantities and vice versa. A hybrid equilibrium might still emerge but it does not exist for all degrees of competition.

Our analysis adapts the same game structure as Singh and Vives (1984), namely firms first independently of each other select their strategy variable and compete in the second stage conditional on this choice.[4] Yet, they are only concerned with the deterministic case while we allow for uncertainty of demand.

One paper that deals with the choice of prices versus quantities in a stochastic environment is Klemperer and Meyer (1986). They consider a one-shot duopoly game in which a firm chooses the decision variable and its magnitude at the same time.[5] They show that uncertainty often reduces the number of equilibria compared to the deterministic case. As the game has a simultaneous structure firms do not choose the mode of competition. Instead each firm acts as a monopolist given its expected residual demand curve.[6] Therefore in their framework the competitive advantage of quantities is not present. Our analysis shows that there is

[2]This seems to be in line with empirical research. For example, Aiginger (1999) asked managers of 930 manufacturing firm in Austria if they select prices or quantities as their decision variable. Roughly, 2/3 charge prices and 1/3 set quantities.

[3]Here the result of the standard asset pricing model (see e.g. Cochrane (2004)), although with a completely different intuition, carries over. If there is more than one shock the covariance becomes the important variable.

[4]Recently, Tasnadi (2006) analyzed a more general model with $n$ firms. He finds that even there the only equilibrium is that all firms select quantities if firms are not capacity constrained.

[5]For a deterministic model with the same game structure but a general number of firms, see Qin and Stuart (1997).

[6]An analysis that considers the monopoly case with a general demand function is provided by Reis (2006).

a trade-off between uncertainty and the degree of competition and derives conditions under which either strategy variable's comparative advantage dominates.[7]

The rest of the paper is organized as follows. Section 2 sets out the model. In Section 3 we solve for the subgame perfect equilibrium in case of a shock affecting the slope. In Section 4 we extend the model by incorporating a shock to the intercept. Section 5 concludes.

# 2 The Model

We consider a duopoly with differentiated products. Firms face the linear demand system

$$p_i = \alpha - \frac{\beta}{\theta} q_i - \frac{\gamma}{\theta} q_j, \tag{1}$$

$$p_j = \alpha - \frac{\beta}{\theta} q_j - \frac{\gamma}{\theta} q_i, \tag{2}$$

with $\alpha > 0$, and $\beta > \gamma \geq 0$. When $\gamma \rightarrow \beta$, products are perfect substitutes whereas with $\gamma = 0$ they are independent. $\theta$ is a random variable and without loss of generality we set $E[\theta] = 1$ and $E[1/\theta] = z$.[8] To avoid unnecessary complications we require the support of the shock to be sufficiently small such that no equilibria emerge in which a price-setting firm sells a negative quantity or a quantity-setting firm receives a negative price. Further we assume that firms have zero marginal costs.[9]

Competition between firms takes the form of a two-stage game. In stage 1 firms simultaneously choose their strategy variables. Each firm observes the other firm's choice and competes in stage 2 contingent on the chosen strategy variables. Afterwards the shock realizes, markets clear and profits accrue. So after the first stage firms are committed to their strategy variable and cannot change it thereafter. We solve for the subgame perfect equilibrium.

[7]For a paper that analyzes the incentives of a social planner to regulate prices or quantities in the presence of demand uncertainty, see Weitzman (1974). Finkelshtain and Kislev (1997) deal with the same issue but instead of a social planner consider political parties and lobbying groups.

[8]Since $E[\theta] = 1$ we have by Jensen's inequality that $z > 1$ if the variance of $\theta$, $\sigma_\theta$, is bigger than zero and that $z$ is increasing in $\sigma_\theta$.

[9]As Singh and Vives (1984) show, the analysis would not change if firms faced positive constant marginal costs $c$ because this would only lower the effective intercept from $\alpha$ to $a = \alpha - c$.

# 3 Solution to the Model

## 3.1 The Second Stage

First, suppose that both firms set prices as their strategy variable. Solving equation (1) and (2) for $q_i$ and $q_j$ gives firm i's expected demand curve, $q_i = \frac{\theta(\alpha(\beta-\gamma)-\beta p_i+\gamma p_j)}{\beta^2-\gamma^2}$. Thus in the second stage firm $i$ maximizes its expected profit by choosing

$$\max_{p_i} \quad E\left[p_i\left(\frac{\theta(\alpha(\beta-\gamma)-\beta p_i+\gamma p_j)}{\beta^2-\gamma^2}\right)\right].$$

Since $E[\theta]=1$ this reduces to

$$\max_{p_i} \quad p_i\left(\frac{\alpha(\beta-\gamma)-\beta p_i+\gamma p_j}{\beta^2-\gamma^2}\right).$$

Solving the maximization problems of both firm $i$ and $j$ yields equilibrium prices of $p_i^\star = p_j^\star = p^\star = \frac{(\beta-\gamma)\alpha}{2\beta-\gamma}$. Therefore each firm's expected profit is equal to

$$\Pi_i^{p^\star p^\star} = \frac{\alpha^2(\beta-\gamma)\beta}{(\beta+\gamma)(2\beta-\gamma)^2}. \tag{3}$$

Next, suppose that both firms set quantities as their strategy variable. From (1) in the second stage firm $i$ maximizes its expected profit by choosing

$$\max_{q_i} \quad E\left[q_i\left(\alpha-\frac{\beta q_i+\gamma q_j}{\theta}\right)\right].$$

Since $E[\frac{1}{\theta}]=z$ this is equivalent to

$$\max_{q_i} \quad q_i(\alpha - z(\beta q_i+\gamma q_j)).$$

Solving the maximization problems of both firms yields equilibrium quantities of $q_i^\star = q_j^\star = q^\star = \frac{\alpha}{z(2\beta+\gamma)}$ and an expected profit of

$$\Pi_i^{q^\star q^\star} = \frac{\alpha^2\beta}{z(2\beta+\gamma)^2} \tag{4}$$

for each firm.

Lastly, if firm $i$ chooses a price while firm $j$ sets a quantity, the demand curve of firm $i$ is $q_i = \frac{\alpha\theta - \theta p_i - \gamma q_j}{\beta}$ and the inverse demand curve of firm $j$ is $p_j = \frac{(\alpha\theta - q_j(\beta+\gamma))(\beta-\gamma)+\gamma\theta p_i}{\beta\theta}$. Setting up the profit functions, maximizing and solving for the equilibrium, yields a price of $p' = \frac{\alpha^2(\beta-\gamma)(2z(\beta+\gamma)-\gamma)}{4z(\beta^2-\gamma^2)+\gamma^2}$ and a quantity of $q' = \frac{\alpha(2\beta-\gamma)}{4z(\beta^2-\gamma^2)+\gamma^2}$. The expected profit of the price setting firm is

$$\Pi_i^{p'q'} = \frac{(\beta-\gamma)^2(2z(\beta+\gamma)\alpha - \alpha\gamma)^2}{(4z(\beta^2-\gamma^2)+\gamma^2)^2\beta}, \tag{5}$$

while the expected profit of the quantity setting firm is

$$\Pi_j^{q'p'} = \frac{z\alpha^2(\beta^2-\gamma^2)(2\beta-\gamma)^2}{(4z(\beta^2-\gamma^2)+\gamma^2)^2\beta}. \tag{6}$$

Before we continue with the analysis we introduce some notation. Conditional on firm $j$ setting a quantity the difference in profits of firm $i$ between setting a price or a quantity is defined as

$$F(\gamma, z) := \Pi_i^{p'q'} - \Pi_i^{q^\star q^\star}.$$

If firm $j$ sets a price this difference is defined as

$$G(\gamma, z) := \Pi_i^{p^\star p^\star} - \Pi_i^{q'p'}.$$

## 3.2 The First Stage

As spelled out before, if the game is deterministic ($\sigma_\theta = 0$) it is a dominant strategy for firms to set quantities in the first stage since they induce a lower degree of competition. This can be easily checked since for all $\gamma \in (0, \beta)$, $F(\gamma, 1) < 0$ and $G(\gamma, 1) < 0$. In the next Lemma, we prove that this is no longer true in case of uncertainty.

**Lemma** For any $\gamma \in (0, \beta)$ there exists a unique $z$, which is labelled $z'(\gamma)$, for which

$$F(\gamma, z'(\gamma)) = 0, \tag{7}$$

and exactly one $z$, labelled $z''(\gamma)$, for which

$$G(\gamma, z''(\gamma)) = 0. \tag{8}$$

If $\gamma = 0$, there exists no $z'$ and no $z''$.

**Proof** The profit functions are rational functions defined on the domain $\gamma \in [0, \beta)$, $z > 1$. Thus $F(\gamma, z')$ and $G(\gamma, z'')$ are continuous in $\gamma$ and $z$ and at least once differentiable.

For an arbitrary $\gamma \in (0, \beta)$

$$\begin{aligned} \lim_{z\to 1} F(\gamma, z) &= \frac{\alpha^2\gamma^3(6\gamma^2\beta - 8\beta^3 + \gamma^3)}{(4\beta^2 - 3\gamma^2)^2\beta(2\beta+\gamma)^2} < 0, \text{ and} \\ \lim_{z\to\infty} F(\gamma, z) &= \frac{\alpha^2}{4\beta} > 0. \end{aligned}$$

Since

$$\frac{\partial F(\gamma, z)}{\partial z} = \frac{4\alpha^2(\beta-\gamma)^2(2z(\beta+\gamma)-\gamma)(\gamma+\beta)\gamma(2\beta-\gamma)}{(4z(\beta^2-\gamma^2)+\gamma^2)^3\beta} + \frac{\alpha^2\beta}{(2\beta+\gamma)^2 z^2} > 0, \tag{9}$$

we have shown that there exists a unique $z'$ for which $F(\gamma, z) = 0$ holds. Note that $\frac{\partial F(\gamma,z)}{\partial z}$ is continuous in $\gamma$ and $z$.

Now we turn to the existence and uniqueness of $z''$. For an arbitrary $\gamma \in (0, \beta)$

$$\begin{aligned} \lim_{z\to 1} G(\gamma, z) &= -\frac{(\beta-\gamma)\alpha^2\gamma^3(8\beta^3 - 6\gamma^2\beta + \gamma^3)}{(\beta+\gamma)(2\beta-\gamma)^2(4\beta^2-3\gamma^2)^2\beta} < 0, \text{ and} \\ \lim_{z\to\infty} G(\gamma, z) &= \frac{\beta(\beta-\gamma)\alpha^2}{(\beta+\gamma)(2\beta-\gamma)^2} > 0. \end{aligned}$$

Since

$$\frac{\partial G(\gamma, z)}{\partial z} = \frac{\alpha^2(2\beta-\gamma)^2(\beta^2-\gamma^2)(4z(\beta^2-\gamma^2)-\gamma^2)}{(4z(\beta^2-\gamma^2)+\gamma^2)^3\beta}, \tag{10}$$

which is negative for $z < \tilde{z} := \frac{\gamma^2}{4(\beta^2-\gamma^2)}$ and positive for $z > \tilde{z}$, we have shown that there exists a unique $z'' > \tilde{z}$ for which $G(\gamma, z) = 0$ is fulfilled. Note that

$\frac{\partial G(\gamma,z)}{\partial z} \mid_{z=z''} > 0$ and that this derivative is continuous in $\gamma$ and $z$.

If $\gamma = 0$, then $F(0,z) = G(0,z) = \frac{\alpha^2(z-1)}{4\beta z}$. Obviously there exists no $z'$ and no $z''$. ■

We have shown that for every $\gamma$ there exists a unique $z$, namely $z'(\gamma)$, such that firm $i$ is indifferent between charging a price or setting a quantity conditional on firm $j$ setting a quantity and that there is another $z$, namely $z''(\gamma)$ such that firm $i$ is indifferent conditional on firm $j$ charging a price. We are now interested in what happens in the neighborhood of the $\gamma$- $z'(\gamma)$-combinations and the $\gamma$-$z''(\gamma)$-combinations. Specifically, if we are on the $\gamma$-$z'(\gamma)$-curve (or the $\gamma$-$z''(\gamma)$-curve) and change one variable while keeping the other one constant, does price or quantity setting become the favorable choice for a firm? In this sense, our next result highlights the trade-off between uncertainty and the degree of substitutability. While a higher amount of risk (higher $z$) favors price setting, a higher degree of competition (higher $\gamma$) favors quantity setting.

**Proposition 1** Along the combinations of $\gamma$ and $z$ for which $F(\gamma,z) = 0$ and $G(\gamma,z) = 0$, an increase in the size of the shock and a decrease in the degree of substitutability increase the comparative advantage of charging prices.

**Proof**

Since (9) is globally strictly positive it is also strictly positive if this derivative is evaluated along the $(\gamma,z)$-combinations for which $F(\gamma,z) = 0$. Thus $\left.\frac{\partial F(\gamma,z)}{\partial z}\right|_{z=z'} > 0$.

In the following we show that the derivative of $F(\gamma,z)$ with respect to $\gamma$ is negative it it is evaluated at $z'$.

Differentiating $F(\gamma,z)$ with respect to $\gamma$ yields:

$$-\frac{2\alpha^2(\beta-\gamma)(2z(\beta+\gamma)-\gamma)(4z(\beta^2-\gamma\beta+\gamma^2)-\gamma^2)}{(4z(\beta^2-\gamma^2)+\gamma^2)^3} + \frac{\alpha^2\beta}{(2\beta+\gamma)^3 z},$$

which is continuous in $\gamma$ and $z$. Evaluating $\frac{\partial F(\gamma,z)}{\partial \gamma}$ at $z'$ yields:

$$\frac{2\alpha^2(\beta-\gamma)(2z'(\beta+\gamma)-\gamma)}{(4z'(\beta^2-\gamma^2)+\gamma^2)^3\beta(2\beta+\gamma)}\,\phi(\gamma,z'), \tag{11}$$

with

$$\phi(\gamma, z) = 8(\beta^2 - \gamma^2)^2 z^2 + 2(\beta^2\gamma^2 - 4\beta^4 - 3\gamma^4)z + \gamma^2(\gamma^2 + 2\beta^2).$$

Since the first factor of (11) is strictly bigger than zero the sign of the derivative is determined by the value of $\phi(\gamma, z)$ at $z'$. Since $\phi(\gamma, z')$ is a quadratic function in $z$ with a positive leading term, it is convex and has two real roots. The one that involves values of $z > 1$ is denoted by $\hat{z}$, where

$$\hat{z} = \frac{4\beta^4 - \beta^2\gamma^2 + 3\gamma^4 + \chi}{8(\beta^2 - \gamma^2)^2},$$

with

$$\chi = \sqrt{49\beta^4\gamma^4 - 24\beta^6\gamma^2 - 6\beta^2\gamma^6 + 16\beta^8 + \gamma^8}.$$

Since $49\beta^4\gamma^4 - 24\beta^6\gamma^2 - 6\beta^2\gamma^6 + 16\beta^8 + \gamma^8$ has no real root it is either always positive or negative for all $\gamma \in [0, \beta)$. Evaluating $\chi^2$ for $\gamma = \frac{\beta}{2}$ yields that it is approximately $13\beta^8$ and so it is always positive. Thus $\chi$ and thereby $\hat{z}$ are well defined.

In the following we compare $\hat{z}$ with $z'$ and use the fact that $F(\gamma, z') = 0$. Evaluating $F(\gamma, z)$ at an arbitrary $\gamma \in (0, \beta)$ and the corresponding $\hat{z}$ yields:

$$\frac{(4\gamma^3\beta - \gamma^4 + 3\gamma^2\beta^2 - 4\gamma\beta^3 + 4\beta^4 + \chi)^2\alpha^2}{4\beta(4\beta^4 + \gamma^2\beta^2 + \gamma^4 + \chi)^2} - \frac{8\alpha^2\beta(\beta^2 - \gamma^2)^2}{(2\beta + \gamma)^2(4\beta^4 - \gamma^2\beta^2 + 3\gamma^4 + \chi)}.$$

To see that $F(\gamma, \hat{z}) > 0 \,\forall \gamma \in (0, \beta)$ we need to rearrange this condition in the following way:

$$\gamma^2\varphi_1\Big(\chi(36\beta^6\gamma^2 + 40\gamma^3\beta^5 + 47\beta^4\gamma^4 + 96\beta^8 + 64\gamma\beta^7 + 40\gamma^5\beta^3 + 2\beta^2\gamma^6 - \gamma^8)$$
$$+ 960\gamma^4\beta^8 + 123\gamma^8\beta^4 + 24\gamma^9\beta^3 + \gamma^{12} + \varphi_2 - \varphi_3\Big),$$

with

$$\varphi_1 = \frac{4\alpha^2}{\beta(\gamma^4 + \beta^2\gamma^2 + 4\beta^4 + \chi)^2(2\beta + \gamma)^2(4\beta^4 - \beta^2\gamma^2 + 3\gamma^4 + \chi)},$$
$$\varphi_2 = 600\beta^7\gamma^5 + 80\beta^5\gamma^7 + 256\beta^{11\gamma} + 384\beta^{12},$$
$$\varphi_3 = 127\beta^6\gamma^6 + 21\beta^2\gamma^{10} + 240\beta^{10}\gamma^2 + 96\beta^9\gamma^3.$$

Obviously $\gamma = 0$ is one of the real roots of $F(\gamma, \hat{z})$. Now we need to show that it is the only one. Since $\varphi_1 > 0$ and

$$\varphi_2 > 127\beta^7\gamma^5 + 21\beta^5\gamma^7 + 240\beta^{11}\gamma + 96\beta^{12} > \varphi_3,$$

for $\gamma \in (0, \beta)$ we have shown, that $F(\gamma, \hat{z})$ has no real root for $\gamma \in (0, \beta)$. Furthermore $F(\gamma, \hat{z}) > 0 \, \forall \, \gamma \in (0, \beta)$. As (7) is increasing in $z$, $z' < \hat{z}$ for every $\gamma \in (0, \beta)$. Thereby $\phi(\gamma, z')$ and thereby the derivative of the left hand side of (7) with respect to $\gamma$ are negative.

Now we turn to the function $G(\gamma, z) = 0$. If (10) is evaluated at $z''$ it is strictly positive, since $z'' > \tilde{z}$. Thus $\left.\frac{\partial G(\gamma,z)}{\partial z}\right|_{z=z''} > 0$. In the following we show that $\left.\frac{\partial G(\gamma,z)}{\partial \gamma}\right|_{z=z''} < 0$.

Differentiating $G(\gamma, z)$ with respect to $\gamma$ yields:

$$2\alpha^2 \left( \frac{(2\beta - \gamma)z(4z(\beta^3 - 2\beta^2\gamma - \beta\gamma^2 + 2\gamma^3) - \gamma(2\gamma^2 + \beta\gamma + 4\beta^2))}{(4z(\beta^2 - \gamma^2) + \gamma^2)^3} - \frac{\beta(\beta^2 - \gamma\beta + \gamma^2)}{(\beta + \gamma)^2(2\beta - \gamma)^3} \right),$$

which is continuous in $\gamma$ and $z$. Evaluating $\frac{\partial G(\gamma,z)}{\partial \gamma}$ at $z''$ yields:

$$\frac{2\alpha^2(2\beta - \gamma)(\beta + \gamma)\gamma}{(4z''(\gamma)(\beta^2 - \gamma^2) + \gamma^2)^3\beta} \psi(\gamma, z''), \tag{12}$$

with

$$\psi(\gamma, z) = ((4z - 1)(6\beta - \gamma)\gamma - (5z - 1)4\beta^2).$$

Since the first factor in (12) is bigger than zero for all $\gamma \in (0, \beta)$ the sign of this derivative is negative, if $\psi(\gamma, z'')$ is negative.

In order to check the sign of $\psi(\gamma, z'')$ we use the fact, that $G(\gamma, z)$ evaluated at $z''$ fulfills the requirements of the Implicit Function Theorem. Thus we solve (8) explicitly for $z''(\gamma)$, which is given by:

$$\frac{\kappa^2 + 8\beta^2\gamma^2(\beta - \gamma) + (2\beta - \gamma)^2\kappa\sqrt{\beta + \gamma}}{32\beta^2(\beta - \gamma)^2(\beta + \gamma)}, \tag{13}$$

with

$$\kappa = \sqrt{16\beta^5 - 8\beta^2\gamma(2\beta^2 + 3\beta\gamma - 4\gamma^2) - \gamma^4(7\beta - \gamma)}.$$

Since $16\beta^5 - 8\beta^2\gamma(2\beta^2 + 3\beta\gamma - 4\gamma^2) - \gamma^4(7\beta - \gamma)$ has no real root and since it is approximately $5\beta^5$ for $\gamma = \frac{\beta}{2}$, $\kappa$ and thereby $z''(\gamma)$ are well defined.

Inserting $z''(\gamma)$ into $\psi(\gamma, z)$ yields:

$$-\frac{(2\beta - \gamma)}{8(\beta^2 - \gamma^2)\beta^2}\Big(24\beta^5 + 31\beta^2\gamma^3 + \gamma^5 - 2\gamma\beta(6\beta^3 + 13\beta^2\gamma + 5\gamma^3)$$
$$+ (5\beta - \gamma)(2\beta - \gamma)\kappa\sqrt{\beta + \gamma}\Big).$$

Since

$$24\beta^5 + 31\beta^2\gamma^3 + \gamma^5 - 2\gamma\beta(6\beta^3 + 13\beta^2\gamma + 5\gamma^3) > \kappa^2 > 0,$$

$\psi(\gamma, z'')$ and $\frac{\partial G(\gamma, z'')}{\partial \gamma}$ are negative. ■

This result makes clear that the degree of uncertainty and the degree of substitutability have offsetting effects. An increase in the degree of substitutability makes the market more competitive. Since quantities are the less aggressive strategy variable such an increase makes quantity setting more attractive relative to price setting. The opposite holds true concerning uncertainty. Let us explain this in more detail. First look at the case in which both firms select quantities. As can be seen from the demand system, (1) and (2), the shock enters the pricing equation in a non-linear way. The bigger the variance $\sigma_\theta^2$ of the shock, the larger is $z$. As can be seen from the demand system the expected price is decreasing in $z$. This is illustrated in an example in Figure 1. Here $\theta$ can take on two values, either $\theta_1 = \frac{1}{2}$ or $\theta_2 = \frac{3}{2}$ with equal probability. Thus, although $E[\theta] = 1$, $z = \frac{4}{3} > 1$. the consequence is that $\frac{1}{2}\Big(p_i^\star(\theta_1) + p_i^\star(\theta_2)\Big) < p_i^\star(E[\theta])$.

As is obvious the price decrease following a bad shock ($a$ in Figure 1) is larger than the price increase following a good shock ($b$ in Figure 1) of same size. Thus, as it is evident from $(4)$ firms' profits in the quantity setting case are decreasing in the size of the shock. Next consider the case in which both firms select prices. From (3) it follows that firms' expected profits are not affected by the size of the shock. The reason is that the shock enters the demand equation in a linear way and therefore

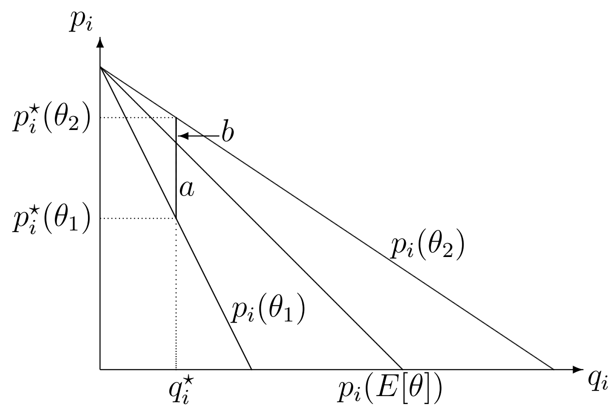


Figure 1: Example with quantity setting

cancels out in expectation. Lastly, in the hybrid case, the shock affects the demand of the price setting firm linearly but it enters non-linearly the inverse demand for the quantity setting firm. The larger is $\sigma_\theta$, the higher is the expected price-decrease that the quantity setting firm experiences if it produces a higher amount. As a consequence it produces less, thereby leaving a larger residual demand curve to the price setting firm which in turn reacts by setting a higher price. Thus, as can be seen from (5) the profit of the price setting firm is increasing in the size of the shock. The profit of the quantity setting firm normally decreases in the size of the shock.[10] This highlights that a higher amount of uncertainty favors the choice of prices while a higher degree of substitution favors quantities.[11] A direct consequence of Proposition 1 is contained in Corollary 1.

**Corollary** 1 Both functions $F(\gamma, z'(\gamma)) = 0$ and $G(\gamma, z'(\gamma)) = 0$ are strictly increasing on the $\gamma$-$z$-plane.

**Proof**

Note that if $F(\gamma, z)$ is evaluated at $z'$ and if $G(\gamma, z)$ is evaluated at $z''$, then the requirements of the Implicit Function Theorem are fulfilled. Since $F_\gamma(\gamma, z'(\gamma)) < 0$,

[10]The exception is if $\gamma > 2\beta\sqrt{\frac{z}{1+4z}}$. In this case the price increase of the other firm overturns the negative effect of the shock and the profit of the quantity setting firm increases in $z$.

[11]It should mentioned that this line of reasoning does not depend on the way we introduced uncertainty in the demand system. If, for example, the demand system would be $p_i = \alpha - \beta\theta q_i - \gamma\theta q_j$, $i, j \in \{1, 2\}$, $i \neq j$, the result would be the same. In that case the shock would have no influence on expected profits if both firms set quantities but instead would increase expected profits if both firms select prices.

$G_\gamma(\gamma, z''(\gamma)) < 0$, $F_z(\gamma, z'(\gamma)) > 0$, and $G_z(\gamma, z''(\gamma)) > 0$ for $\gamma \in (0, \beta)$ it follows from the Implicit Function Theorem that

$$\frac{dz'(\gamma)}{d\gamma} = -\frac{F_\gamma(\gamma, z'(\gamma))}{F_z(\gamma, z'(\gamma))} > 0, \tag{14}$$

$$\frac{dz''(\gamma)}{d\gamma} = -\frac{G_\gamma(\gamma, z''(\gamma))}{G_z(\gamma, z''(\gamma))} > 0. \tag{15}$$

■

It remains to determine the position of the $F(\gamma, z'(\gamma))$ and $G(\gamma, z'(\gamma))$ curve. This is done in the next Proposition which characterizes the subgame perfect equilibrium of the two stage game under uncertainty.[12]

**Proposition** 2 The subgame perfect equilibrium outcome of the two-stage-game is the following:
Both firms select a quantity in the first stage if

$$z < z'(\gamma).$$

One firms selects a price and the other firm a quantity in the first stage if

$$z''(\gamma) > z \geq z'(\gamma).$$

Both firms select a price in the first stage if

$$z \geq z''(\gamma).$$

$z'(\gamma)$ and $z''(\gamma)$ are defined implicitly by (7) and (8).

**Proof** From the last proposition we know that if $z < z'(\gamma)$ both firms prefer to set quantities while if $z \geq z''(\gamma)$ both firms prefer to set prices. We will now show that $z''(\gamma) > z'(\gamma)$ for all $\gamma \in (0, \beta)$. Consider an arbitrary $\gamma \in (0, \beta)$ and the

[12]We do only characterize the equilibrium in pure strategies. The reason is that it is not interpretable and unrealistic that a firm mixes between a price and a quantity.

associated $z''(\gamma)$ for which $(8)$ holds. If $(7)$ is evaluated at that $z''(\gamma)$ this condition becomes:

$$\frac{4\alpha^2\gamma^2(2\beta-\gamma)^2}{\beta(2\beta+\gamma)^2}\lambda_1\Big(\kappa\sqrt{\beta+\gamma}(2\beta-\gamma)\lambda_2+\lambda_3\Big), \tag{16}$$

with

$$\begin{aligned}\lambda_1 = \Big(&(16\beta^3(\beta^2-\beta\gamma-\gamma^2)+\gamma^3(24\beta^2-7\gamma\beta+\gamma^2)+(2\beta-\gamma)^2\kappa\sqrt{\beta+\gamma})\\ &(16\beta^2(\beta^3-\beta^2\gamma+\gamma^3)+\gamma^2(\gamma^3-7\gamma^2\beta-8\beta^3)+(2\beta-\gamma)^2\kappa\sqrt{\beta+\gamma})^2\Big)^{-1},\end{aligned}$$

$$\lambda_2 = (288\beta^7-304\beta^6\gamma-176\beta^5\gamma^2+304\beta^4\gamma^3-70\beta^3\gamma^4-13\beta^2\gamma^5+8\beta\gamma^6-\gamma^7),$$

and

$$\begin{aligned}\lambda_3 =&\gamma^{11}-13\beta\gamma^{10}+67\beta^2\gamma^9-95\beta^3\gamma^8-592\beta^4\gamma^7+2832\beta^5\gamma^6-3536\beta^6\gamma^5\\ &-1920\beta^7\gamma^4+7168\beta^8\gamma^3-3072\beta^9\gamma^2-2560\beta^{10}\gamma+1792\beta^{11}.\end{aligned}$$

Now we need to determine the sign of $\lambda_1$, $\lambda_2$, and $\lambda_3$. Since

$$\begin{aligned}16\beta^3(\beta^2-\beta\gamma-\gamma^2)+\gamma^3(24\beta^2-7\gamma\beta+\gamma^2) &\geq \kappa^2>0, \text{and}\\ 16\beta^2(\beta^3-\beta^2\gamma+\gamma^3)+\gamma^2(\gamma^3-7\gamma^2\beta-8\beta^3) &\geq \kappa^2>0,\end{aligned}$$

$\lambda_1$ is strictly bigger than zero. $\lambda_2$ and $\lambda_3$ have no real root. Since

$$\begin{aligned}\lambda_2\Big(\frac{\beta}{2}\Big) &\approx 125\beta^7>0, \text{ and}\\ \lambda_3\Big(\frac{\beta}{2}\Big) &\approx 449\beta^{11}>0,\end{aligned}$$

both expressions are strictly bigger than zero. This implies that $(16)$ is bigger than zero. Since the left hand side of $(7)$ is increasing in the size of the shock, $z$ has to be smaller than $z''(\gamma)$ in order for this condition to hold.

>From Proposition 1 we know that if $z<z''(\gamma)$ and one firms sets a price the other firm prefers to set a quantity while if $z\geq z'(\gamma)$ and one firm sets a quantity the other one prefers to set a price. Thus, for $z''(\gamma)>z\geq z'(\gamma)$ a hybrid equilibrium exists.

■

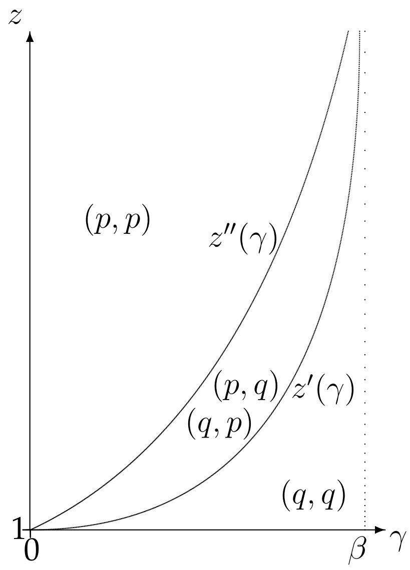


Figure 2: Equilibrium of the two-stage game

The different equilibrium regions are displayed in Figure 2.

If uncertainty is high relative to the degree of substitution both firms select prices while they both select quantities if the opposite holds true.[13] But there always emerges a hybrid equilibrium in which one firm charges a prices and the other one sets a quantity. The intuition behind this result is the following: For a given degree of substitutability the degree of competition is always fiercest if both firms charge prices while it is lowest when both set quantities. As a consequence, the size of the shock, that induces both firms to play the price game must be higher than the one that makes a firm indifferent between setting a price or setting a quantity conditional on the other firm setting a quantity. Thus, there always exist a range of $z$ such that an equilibrium emerges in which firms set different strategy variables.

We have restricted our attention to the case in which products are substitutes, i.e. $\gamma \geq 0$. Here we note briefly that if products are complements, i.e. $\gamma < 0$, then it is a dominant strategy for both firms to set a price irrespective of the degree of uncertainty. The reason for this result is that both the competition and the uncer-

[13] If products are nearly perfect substitutes ($\gamma \rightarrow \beta$), then $z'(\gamma) \rightarrow \infty$ and quantities are the preferred choice for every $z$.

tainty effect favor prices. Firstly, as Singh and Vives (1984) have shown, if products are complements setting prices is the dominant strategy in a deterministic environment. Secondly, the comparative advantage of setting a price with respect to uncertainty is still present since this effect is independent from the products' relationship, namely if they are substitutes or complements. Consequently the well-known duality relationship firstly discovered by Sonnenschein (1968) does no longer hold under uncertainty.

# 4 Two-Dimensional Uncertainty

So far we only considered a shock affecting the slope of the demand curve. This means that a firm knows the range of its reservation prices but does not know the distribution. But in reality even the range of reservation prices is uncertain. So we now augment our original model by additionally considering a shock that affects the intercept. The demand system changes to

$$p_i = \alpha + \epsilon - \frac{\beta}{\theta} q_i - \frac{\gamma}{\theta} q_j, \tag{17}$$

$$p_j = \alpha + \epsilon - \frac{\beta}{\theta} q_j - \frac{\gamma}{\theta} q_i, \tag{18}$$

where $\epsilon$ is a random variable with $E[\epsilon] = 0$, Var$[\epsilon] > 0$. The shock on the intercept might be correlated with the shock affecting the slope and we term the covariance between the two shocks by $\sigma_{\theta\epsilon}$. As before we solve for the subgame perfect equilibrium and start with the second stage.

Proceeding in the same way as before yields an equilibrium profit of

$$\Pi^{p^{\star\star}p^{\star\star}} = \frac{(\alpha + \sigma_{\theta\epsilon})^2(\beta - \gamma)\beta}{(\beta + \gamma)(2\beta - \gamma)^2}$$

in the case when both firms select a price as the strategy variable in the first stage and an equilibrium profit of

$$\Pi^{q^{\star\star}q^{\star\star}} = \frac{\alpha^2 \beta}{z(2\beta + \gamma)^2},$$

in case both firms select a quantity as the strategic variable in the first stage. In case firms select different strategy variables the price setting firm $i$ receives an equilibrium profit of

$$\Pi_i^{p^{\star\star}q^{\star\star}} = \frac{(\beta-\gamma)^2(2z(\beta+\gamma)(\alpha+\sigma_{\theta\epsilon})-\alpha\gamma)^2}{(4z(\beta^2-\gamma^2)+\gamma^2)^2\beta},$$

while the quantity setting firm $j$ receives an equilibrium profit of

$$\Pi_j^{q^{\star\star}p^{\star\star}} = \frac{(\beta^2-\gamma^2)z(\alpha(2\beta-\gamma)+\gamma\sigma_{\theta\epsilon})^2}{(4z(\beta^2-\gamma^2)+\gamma^2)^2\beta}.$$

We can now move on to the first stage, in which firms choose their strategy variable. We focus on the case of a nonnegative covariance and only briefly discuss the results for a negative covariance. The complete characterization of the equilibrium for the case of a negative covariance can be found in the Appendix. The reason for that is twofold. Firstly, the case of a nonnegative covariance is much more realistic. Suppose demand conditions are good. If some consumers are willing to pay a high price for the good (positive shock on the intercept) it is likely that that the market becomes larger as well (positive shock on the slope) and vice versa if demand conditions are bad. On the other hand it is hard to imagine a market in which a positive shock on the intercept is coupled with the expectation of a decreasing market size. Secondly, the analysis of a negative covariance proceeds along very similar lines to the one of a nonnegative covariance and for the sake of exposition we focus on the latter.

**Proposition** 3 The subgame perfect equilibrium outcome of the two-stage-game assuming a nonnegative covariance is the following:

If $\sigma_{\theta\epsilon} = 0$ the outcome is the same as in the case with no additive shock.

If $\sigma_{\theta_\epsilon} > 0$ the outcome of the game is the following:
Both firms select a quantity in the first stage if

$$\gamma \in [\gamma', \beta) \text{ and } \sigma_{\theta\epsilon}(\gamma) < \sigma^\star_{\theta\epsilon}(\gamma),$$

One firms selects a price and the other firm a quantity in the first stage if

$$\gamma \in [\gamma'', \beta) \text{ and } \sigma^{\star\star}_{\theta\epsilon}(\gamma) > \sigma_{\theta\epsilon}(\gamma) \geq \sigma^{\star}_{\theta\epsilon}(\gamma).$$

Both firms select a price in the first stage if

$$\gamma \in (0, \gamma^{+}) \text{ and } \sigma_{\theta\epsilon}(\gamma) \geq \sigma^{\star\star}_{\theta\epsilon}(\gamma).$$

**Proof** First look at the case in which firm $j$ sets a quantity. Firm $i$ is indifferent between setting a price or a quantity if

$$\frac{(\beta-\gamma)^2(2z(\beta+\gamma)(\alpha+\sigma_{\theta\epsilon})-\alpha\gamma)^2}{(4z(\beta^2-\gamma^2)+\gamma^2)^2\beta} = \frac{\alpha^2\beta}{z(2\beta+\gamma)^2} \tag{19}$$

The threshold covariance $\sigma^{\star}_{\theta\epsilon}(\gamma)$ above which firm $i$ prefers to charge a price is

$$\sigma^{\star}_{\theta\epsilon}(\gamma) = \frac{\alpha(\sqrt{z\beta^2(4z(\beta^2-\gamma^2)+\gamma^2)^2}) - z(2\beta+\gamma)(\beta-\gamma)(2z(\beta+\gamma)-\gamma)}{2z^2(\beta^2-\gamma^2)(2\beta+\gamma)}.^{14} \tag{20}$$

The threshold covariance is nonnegative if $\gamma \geq \gamma'$, where $\gamma'$ is implicitly defined by $f(\gamma') = 0$ where $f(\gamma)$ is the numerator of (20). Since $f(\gamma)$ is strictly convex in $\gamma$ with $f(0) < 0$ and $f(\beta) > 0$, it follows that $\gamma' \in (0, \beta)$ is unique.

Now suppose firm $j$ sets a price. Then firm $i$ is indifferent between choosing a price or a quantity if

$$\frac{(\alpha+\sigma_{\theta\epsilon})^2(\beta-\gamma)\beta}{(\beta+\gamma)(2\beta-\gamma)^2} = \frac{(\beta^2-\gamma^2)z(\alpha(2\beta-\gamma)+\gamma\sigma_{\theta\epsilon})^2}{(4z(\beta^2-\gamma^2)+\gamma^2)^2\beta}. \tag{21}$$

[14] Since (19) is quadratic in $\sigma_{\theta\epsilon}$ there exists a second threshold covariance which is strictly negative and thus can be neglected.

The threshold covariance $\sigma^{\star\star}_{\theta\epsilon}(\gamma)$ above which firm $i$ prefers to charge a price is[15]

$$\sigma^{\star\star}_{\theta\epsilon}(\gamma) = \frac{\alpha\Big(\gamma(\beta+\gamma)(2\beta^2(4\beta^2-6\beta\gamma+\gamma^2)+\gamma^3(5\beta-\gamma))z-\beta^2(16z^2(\beta^2-\gamma^2)^2+\gamma^4)\Big)}{16\beta^2(\beta^2-\gamma^2)^2z^2+\gamma^2(\beta+\gamma)(4\beta^3-8\beta^2\gamma+3\beta\gamma^2-\gamma^3)z+\beta^2\gamma^4} + \frac{2\alpha\sqrt{z}\beta(2\beta-\gamma)(\beta^2-\gamma^2)(4z(\beta^2-\gamma^2)+\gamma^2)}{16\beta^2(\beta^2-\gamma^2)^2z^2+\gamma^2(\beta+\gamma)(4\beta^3-8\beta^2\gamma+3\beta\gamma^2-\gamma^3)z+\beta^2\gamma^4}. \tag{22}$$

Now we determine the range in which $\sigma^{\star\star}_{\theta\epsilon}(\gamma) \geq 0$. Let $\gamma''$ and $\gamma^+$ be implicitly defined by $g(\gamma'') = 0$ and $h(\gamma^+) = 0$ where $g(\gamma)$ is the numerator of (22) and $h(\gamma)$ is the denominator of (22).

First, we show, that $\gamma'', \gamma^+ \in (0, \beta)$ are unique. Since $g(\gamma)$ decreases in $\gamma$ for $\gamma$ smaller than some $\gamma^\star \in (0, \beta)$ and strictly increases thereafter with $g(0) < 0$ and $g(\beta) > 0$ it follows that $\gamma'' \in (0, \beta)$ is unique. Similarly, $h(\gamma)$ increases in $\gamma$ for $\gamma$ smaller than some $\gamma^{\star\star} \in (0, \beta)$ and strictly decreases thereafter with $h(0) > 0$ and $h(\beta) < 0$. Therefore $\gamma^+ \in (0, \beta)$ is unique. Next we show that $\gamma'' < \gamma^+$. Since $\sigma^{\star\star}_{\theta\epsilon}(\gamma)$ is strictly convex in $\gamma$ for $\gamma \in [0, \gamma^+)$ with $\sigma^{\star\star}_{\theta\epsilon}(\gamma) < 0$ if $\gamma = 0$ and $\lim_{\gamma\to\gamma^+} \sigma^{\star\star}_{\theta\epsilon}(\gamma) \to \infty$ it follows that $\gamma'' < \gamma^+$. Thus, the numerator and the denominator of $\sigma^{\star\star}_{\theta\epsilon}(\gamma)$ have the same sign if and only if $\gamma \in (\gamma'', \gamma^+)$. Therefore $\sigma^{\star\star}_{\theta\epsilon}(\gamma) \geq 0$ if $\gamma \in [\gamma'', \gamma^+)$.

Now we show that for $\gamma'' \leq \gamma < \gamma^+$ firm $i$ prefers to set a quantity contingent on firm $j$ choosing a price if $0 \leq \sigma_{\theta\epsilon} < \sigma^{\star\star}_{\theta\epsilon}(\gamma)$. Firm $i$ sets a quantity if the right hand side of (21) is bigger than the left hand side. Subtracting teh right hand side from the left hand side and differentiating this difference twice with respect to $\sigma_{\theta\epsilon}$ yields that it is convex if $h(\gamma) > 0$ which is the case if $\gamma < \gamma^+$. We know that thw difference is zero at $\sigma^{\star\star}_{\theta\epsilon}(\gamma)$ Thus, if $\gamma'' \leq \gamma < \gamma^+$, it follows that for $\sigma_{\theta\epsilon} < \sigma^{\star\star}_{\theta\epsilon}(\gamma)$ the difference is negative and firm $i$ prefers to set a quantity, while for $\sigma_{\theta\epsilon}(\gamma) \geq \sigma^{\star\star}_{\theta\epsilon}(\gamma)$ it prefers a price.

Now consider the case $\gamma > \gamma^+$. This implies that $\sigma^{\star\star}_{\theta\epsilon}(\gamma)$ is negative and the difference in expected profits is concave in $\sigma_{\theta\epsilon}$. Thus, the difference is negative if $\sigma_{\theta\epsilon} \geq \sigma^{\star\star}_{\theta\epsilon}(\gamma)$ and consequently, firm $i$ prefers to set a quantity for every positive

[15] As in the previous case, there exists a second threshold covariance, that is strictly negative and is therefore not relevant. Again an increase in $\sigma_\theta$ causes the left hand side of (22) to increase and its right hand side to decrease.

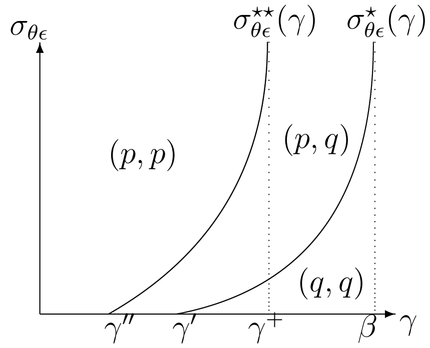


Figure 3: Equilibrium with nonnegative covariance

covariance.

Finally, it is easy to show that $\sigma^{\star\star}_{\theta\epsilon}(\gamma) > \sigma^{\star}_{\theta\epsilon}(\gamma)$ for all $\gamma'' \leq \gamma < \gamma^+$. This implies that $\gamma' > \gamma''$. ■

The outcome of the game is depicted in Figure 3. There are two main differences to the analysis without a shock on the intercept.

The first is that the important parameter determining the choice of the strategy variable is now the covariance instead of the variance. The reason is that the interplay of the shocks is now crucial for the position of the residual demand curve after realization of the shock. For example, if a firm selects a quantity as its decision variable in the first stage the interplay between the shocks determines the price that it receives. Thus the covariance enters the expected profit and determines the strategy variable chosen in equilibrium.[16]

The second difference is that not for all three kinds of equilibria exist for all values of $\gamma$. Instead, if $\gamma$ is small there exists only an equilibrium in which both firms charge prices while if $\gamma$ is large there exists only the quantity-quantity-equilibrium and the hybrid one. As can be seen from (19) and (21) the reason for that is that the expected profits in the hybrid equilibrium and in the price-price-equilibrium are increasing in the covariance while the expected profit in the quantity-quantity-equilibrium is not affected by the covariance. As a consequence, the range for which the quantity-quantity-equilibrium exists shrinks compared to the case without the shock on the intercept and so this equilibrium does no longer exist if $\gamma$ is

[16]This is somehow reminiscent to many asset pricing models (see e.g. Cochrane (2004)) in which investors base their optimal decisions on the covariance of returns. Although the intuition is completely different, in our context firms base the decision about the their strategy variable also on the covariance between the two shocks.

small. If the degree of competition is high ($\gamma$ close to $\beta$) no price-price-equilibrium exists anymore. This is because a high covariance is more beneficial for firm $i$ if it selects a quantity instead of a price, conditional on firm $j$ selecting a price .

We now briefly turn to the case with a negative covariance. The equilibrium of this case is given in detail in Proposition 4 in the Appendix. Here we only discuss two peculiarities that arise in equilibrium with a negative covariance. The first one is that even if $\gamma$ is close to zero there exists a quantity-quantity-equilibrium for some values of $\sigma_{\theta\epsilon}$, so the superiority of the price-price-equilibrium with uncertainty might vanish. With a negative covariance the shocks affect the slope and the intercept of the expected demand function in different directions. As a consequence, the overall uncertainty is reduced for intermediate values of the covariance. This effect can be so large that only the competitive advantage of quantities remains resulting in the quantity-quantity-equilibrium. Of course, if the covariance becomes smaller (more negative) the shock on the slope dominates again and for small $\gamma$, firms again optimally select the price-price-game.

The second peculiarity that arises with a negative covariance, is the existence of multiple equilibria. For some $\gamma$-$\sigma_{\theta\epsilon}$-combinations both the price-price and the quantity-quantity-equilibrium exist. The intuition behind this result is the following. The magnitude of the covariance that is necessary to induce a firm to charge a price is smaller if the other firm charges a price as well than if it selected a quantity. Thus if firm $j$ charges a price it is optimal for firm $i$ to charge a price as well. But if firm $j$ sets a quantity firm $i$ optimally selects a quantity as well. Thus multiple equilibria exist.

# 5 Conclusion

We show in this paper that the superiority of quantity competition for firms might no longer hold if there is a substantial amount of uncertainty concerning demand conditions. If this uncertainty is high relative to the degree of substitution firms prefer to set prices rather than quantities. Moreover, there also exists a hybrid equilibrium for an intermediate range of uncertainty. We also demonstrated that if there are two kinds of shocks the important variable determining the mode of competition is the covariance between the two shocks. The paper also provides the testable implication that if firms have some degree of choice about their strategy variable, they should tend to choose quantities in industries with relatively stable and certain demand, but choose prices if demand is fluctuating and uncertain.

# 6 Appendix

**Proposition** 4 The subgame perfect equilibrium outcome of the two-stage-game assuming a negative covariance is the following:

If $\gamma \in (0, \check{\gamma}]$ both firms select a quantity in the first stage if $\sigma^{\star\star}_{\theta\epsilon}(\gamma) \geq \sigma_{\theta\epsilon} \geq \widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$. Both firms select a price if $0 > \sigma_{\theta\epsilon} > \sigma^{\star}_{\theta\epsilon}(\gamma)$ or $\sigma_{\theta\epsilon} < \widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma)$. Multiple equilibria occur if $\sigma^{\star}_{\theta\epsilon}(\gamma) > \sigma_{\theta\epsilon} > \sigma^{\star\star}_{\theta\epsilon}(\gamma)$ or $\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma) > \sigma_{\theta\epsilon} > \widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma)$.

If $\gamma \in (\check{\gamma}, \gamma^{+}]$ both firms select a quantity in the first stage if $\sigma^{\star}_{\theta\epsilon}(\gamma) \geq \sigma_{\theta\epsilon} \geq \widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$, whereas both firms select a price if $0 > \sigma_{\theta\epsilon} > \sigma^{\star}_{\theta\epsilon}(\gamma)$ or $\sigma_{\theta\epsilon} < \widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma)$. If $\sigma^{\star\star}_{\theta\epsilon}(\gamma) > \sigma_{\theta\epsilon} > \sigma^{\star}_{\theta\epsilon}(\gamma)$ one firm charges a price whereas the other firm selects a quantity. Multiple equilibria emerge if $\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma) > \sigma_{\theta\epsilon} > \widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma)$.

If $\gamma \in (\gamma^{+}, \hat{\gamma}]$ both firms select a quantity in the first stage if $\sigma^{\star}_{\theta\epsilon}(\gamma) \geq \sigma_{\theta\epsilon} \geq \widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$. Both firms select a price if $\widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma) > \sigma_{\theta\epsilon} > \sigma^{\star\star}_{\theta\epsilon}(\gamma)$. One firm charges a price whereas the other firm selects a quantity if $0 > \sigma_{\theta\epsilon} > \sigma^{\star}_{\theta\epsilon}(\gamma)$ or $\sigma^{\star\star}_{\theta\epsilon}(\gamma) > \sigma_{\theta\epsilon}$. Multiple equilibria emerge if $\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma) > \sigma_{\theta\epsilon} > \widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma)$.

If $\gamma \in (\hat{\gamma}, \beta)$ both firms select a quantity in the first stage if $0 > \sigma_{\theta\epsilon} \geq \widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$ or $\sigma^{\star\star}_{\theta\epsilon}(\gamma) \geq \sigma_{\theta\epsilon} \geq \widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma)$. One firm charges a price whereas the other firm selects a quantity if $\widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma) > \sigma_{\theta\epsilon}$. Multiple equilibria emerge if $\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma) > \sigma_{\theta\epsilon} > \sigma^{\star\star}_{\theta\epsilon}(\gamma)$.

Here $\widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma)$ is defined by $(23)$ and $\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$ is defined by $(24)$.

**Proof**

Since we consider a negative covariance we have to take the negative roots of (19) and (21) into account. The former is given by

$$\widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma) = -\frac{\sqrt{z}\beta(4z(\gamma^2-\beta^2)-\gamma)^2 + (z(2\beta+\gamma)(\beta-\gamma)(2z(\beta+\gamma)-\gamma)\alpha}{2z^2(\beta^2-\gamma^2)(2\beta+\gamma)(\beta-\gamma)}, \qquad (23)$$

and the latter by

$$\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma) = \frac{\alpha\Big(\gamma(\beta+\gamma)(2\beta^2(4\beta^2-6\beta\gamma+\gamma^2)+\gamma^3(5\beta-\gamma))z-\beta^2(16z^2(\beta^2-\gamma^2)^2+\gamma^4)\Big)}{16\beta^2(\beta^2-\gamma^2)^2z^2+\gamma^2(\beta+\gamma)(4\beta^3-8\beta^2\gamma+3\beta\gamma^2-\gamma^3)z+\beta^2\gamma^4} - \frac{2\sqrt{z}\beta(2\beta-\gamma)(\beta^2-\gamma^2)(4z(\beta^2-\gamma^2)+\gamma^2)}{16\beta^2(\beta^2-\gamma^2)^2z^2+\gamma^2(\beta+\gamma)(4\beta^3-8\beta^2\gamma+3\beta\gamma^2-\gamma^3)z+\beta^2\gamma^4}. \tag{24}$$

Note that (24) is defined for $\gamma \in (0,\beta)\setminus\{\gamma^+\}$. Whereas $\sigma^{\star}_{\theta\epsilon}(\gamma) > \widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma)\,\forall\,\gamma\in(0,\beta)$, $\sigma^{\star\star}_{\theta\epsilon}(\gamma) > \widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$ if $\gamma\in(0,\gamma^+)$ and $\sigma^{\star\star}_{\theta\epsilon}(\gamma) < \widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$ if $\gamma\in(\gamma^+,\beta)$.

Now we characterize the equilibrium regions for $\gamma\in(0,\gamma^+)$.

First note that $\lim_{\gamma\to0}\sigma^{\star}_{\theta\epsilon} = \lim_{\gamma\to0}\sigma^{\star\star}_{\theta\epsilon} = -\frac{\alpha(z^2-\sqrt[3]{z^2})}{z^2} < 0$ and $\lim_{\gamma\to0}\widehat{\sigma^{\star}_{\theta\epsilon}} = \lim_{\gamma\to0}\widehat{\sigma^{\star\star}_{\theta\epsilon}} = -\frac{\alpha(\sqrt[3]{z^2}+z^2)}{z^2} < -\frac{\alpha(z^2-\sqrt[3]{z^2})}{z^2} < 0$. So both the bigger and the smaller threshold covariances originate in the same points.

In order to analyze how $\sigma^{\star}_{\theta\epsilon}(\gamma)$ and $\sigma^{\star\star}_{\theta\epsilon}(\gamma)$ are related to each other we subtract the former from the latter. This yields:

$$\frac{\alpha(4z(\beta^2-\gamma^2)+\gamma^2)}{2h(\gamma)(2\beta+\gamma)(\beta^2-\gamma^2)}\Bigg(z(2\beta+\gamma)(\beta-\gamma)(\gamma^2(\beta z(4\beta^2-2\gamma^2+\gamma\beta)+\gamma^3z-\gamma\beta^2) + 4\sqrt{z^3}\beta(\beta-\gamma)(2\beta-\gamma)(\beta+\gamma)^2)-\beta\sqrt{z}h(\gamma)\Bigg). \tag{25}$$

This difference is equal to zero for $\check{\gamma} = \frac{(\sqrt{z}-1)\beta}{\sqrt{z}}$ and smaller than zero for $\gamma < \check{\gamma}$. Thus we have shown that $\sigma^{\star\star}_{\theta\epsilon}(\gamma) \le \sigma^{\star}_{\theta\epsilon}(\gamma)$ if $\gamma \le \check{\gamma}$ and $\sigma^{\star\star}_{\theta\epsilon}(\gamma) > \sigma^{\star}_{\theta\epsilon}(\gamma)$ if $\gamma^+ > \gamma > \check{\gamma}$.

Now we turn to the relation between $\widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma)$ and $\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$. Subtracting the former from the latter yields:

$$\frac{\alpha(4z(\beta^2-\gamma^2)+\gamma^2)}{2h(\gamma)(2\beta+\gamma)(\beta^2-\gamma^2)}\Bigg(z(2\beta+\gamma)(\beta-\gamma)(\gamma^2(\beta z(4\beta^2-2\gamma^2+\gamma\beta)+\gamma^3z-\gamma\beta^2) - 4\sqrt{z^3}\beta(\beta-\gamma)(2\beta-\gamma)(\beta+\gamma)^2)+\beta\sqrt{z}h(\gamma)\Bigg). \tag{26}$$

This difference has no real root for $\gamma \in (0, \gamma^+)$. (26) evaluated at $\gamma = \frac{\beta}{2}$ yields

$$\frac{\alpha\Big(8\sqrt{z} - 20z + 126\sqrt{z^3} - 75z^2 + 432\sqrt{5}(2z+1)\Big)}{30z^2(576z^2 + 15z + 4)} > 0.$$

Thus $\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma) > \widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma)$ for $\gamma \in (0, \gamma^+)$.

Finally we have to check whether $\sigma^{\star}_{\theta\epsilon}(\gamma)$ is bigger or smaller than $\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$. Subtracting the former from the latter yields

$$-\frac{\alpha(4z(\beta^2-\gamma^2)+\gamma^2)}{2h(\gamma)(2\beta+\gamma)(\beta^2-\gamma^2)}\Bigg(z(2\beta+\gamma)(\beta-\gamma)(\gamma^2(\beta z(4\beta^2-2\gamma^2+\gamma\beta)+\gamma^3 z-\gamma\beta^2) \\ +4\sqrt{z^3}\beta(\beta-\gamma)(2\beta-\gamma)(\beta+\gamma)^2)+\beta\sqrt{z}h(\gamma)\Bigg). \tag{27}$$

This difference is obviously negative for $\gamma \in (0, \gamma^+)$. Thus $\sigma^{\star}_{\theta\epsilon}(\gamma)$ is bigger $\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$ for $\gamma \in (0, \gamma^+)$.

After characterizing the threshold covariances the equilibria of the two stage game for $\gamma \in (0, \gamma^+)$ can be determined by the following reasoning. Conditional on firm $j$ setting a quantity, firm $i's$ best response is to set a quantity as well if $\sigma^{\star}_{\theta\epsilon}(\gamma) \geq \sigma_{\theta\epsilon} \geq \widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma)$. If $\sigma_{\theta\epsilon} > \sigma^{\star}_{\theta\epsilon}(\gamma)$ or $\sigma_{\theta\epsilon} < \widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma)$ its best response is to charge a price. Conditional on firm $j$ charging a price, firm $i's$ best response is to charge a price as well if $\sigma_{\theta\epsilon} > \sigma^{\star\star}_{\theta\epsilon}(\gamma)$ or $\sigma_{\theta\epsilon} < \widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$. If $\sigma^{\star\star}_{\theta\epsilon}(\gamma) \geq \sigma_{\theta\epsilon} \geq \widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$ its best response is to set a quantity. The same line of reasoning applies for firm $j$. Combining these facts with each other proves the statements given in the first two paragraphs of the proposition.

Now we characterize equilibrium regions for $\gamma \in (\gamma^+, \beta)$. Note that $h(\gamma) < 0$ for $\gamma > \gamma^+$. First we consider whether the relation between $\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma), \widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma)$, and $\sigma^{\star}_{\theta\epsilon}(\gamma)$ changes for $\gamma \in (\gamma^+, \beta)$. The relation between $\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$ and $\widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma)$ is reflected in (26). This difference has no real root for $\gamma \in (\gamma^+, \beta)$ either. Since $\lim_{\gamma\to\beta}\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma) = -\alpha$ and $\lim_{\gamma\to\beta}\widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma) = -\infty$ this difference converges to infinity. Thus $\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$ is strictly bigger than $\widehat{\sigma^{\star}_{\theta\epsilon}}(\gamma)$ for $\gamma \in (\gamma^+, \beta)$. Next we look at the relation between $\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$ and $\sigma^{\star}_{\theta\epsilon}(\gamma)$ that is reflected in (27). This difference has no real root for $\gamma \in$

$(\gamma^+, \beta)$. Since $\lim_{\gamma\to\beta} \sigma^\star_{\theta\epsilon} = \infty$ this difference converges to $-\infty$. Thus $\sigma^\star_{\theta\epsilon}(\gamma)$ remains to be bigger than $\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$ for $\gamma \in (\gamma^+, \beta)$.

Now we characterize the relation between $\sigma^{\star\star}_{\theta\epsilon}(\gamma)$ and $\widehat{\sigma^\star_{\theta\epsilon}}(\gamma)$. Subtracting the latter from the former yields:

$$\frac{\alpha(4z(\beta^2-\gamma^2)+\gamma^2)}{2h(\gamma)(2\beta+\gamma)(\beta^2-\gamma^2)}\Bigg(z(2\beta+\gamma)(\beta-\gamma)(-\gamma^2(\beta z(4\beta^2-2\gamma^2+\gamma\beta)+\gamma^3 z-\gamma\beta^2) \\ +4\sqrt{z^3}\beta(\beta-\gamma)(2\beta-\gamma)(\beta+\gamma)^2)+\beta\sqrt{z}h(\gamma)\Bigg). \tag{28}$$

Whereas this difference converges to $\infty$ if $\gamma \to \beta$, $\lim_{\gamma\downarrow\gamma^+} \sigma^{\star\star}_{\theta\epsilon}(\gamma) = -\infty$ and $0 > \lim_{\gamma\downarrow\gamma^+} \widehat{\sigma^\star_{\theta\epsilon}}(\gamma) > -\infty$ the difference converges to $-\infty$. Since (28) is a continuous function of $\gamma$ there exists a $\gamma \in (\gamma^+, \beta)$ for which the difference of the threshold covariances is zero. Differentiating (28) with respect to $\gamma$ yields that it is strictly increasing in $\gamma$ for $\gamma > \gamma^+$.[17] Thus $\hat{\gamma}$, that is implicitly defined by $k(\hat{\gamma}) = 0$, where $k(\gamma)$ is the numerator of (28) is unique.

After characterizing the threshold covariances the equilibria of the two stage game for $\gamma \in (\gamma^+, \beta)$ can be determined by the following reasoning. Conditional on firm $j$ setting a quantity, firm $i's$ best response is to set a quantity as well if $\sigma^\star_{\theta\epsilon}(\gamma) \geq \sigma_{\theta\epsilon} \geq \widehat{\sigma^\star_{\theta\epsilon}}(\gamma)$. If $\sigma_{\theta\epsilon} > \sigma^\star_{\theta\epsilon}(\gamma)$ or $\sigma_{\theta\epsilon} < \widehat{\sigma^\star_{\theta\epsilon}}(\gamma)$ its best response is to charge a price. Whereas conditional on firm $j$ charging a price, firm $i's$ best response is to charge a price as well if $\widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma) \geq \sigma_{\theta\epsilon} \geq \sigma^{\star\star}_{\theta\epsilon}(\gamma)$. If $\sigma_{\theta\epsilon} < \sigma^{\star\star}_{\theta\epsilon}(\gamma)$ or $\sigma_{\theta\epsilon} > \widehat{\sigma^{\star\star}_{\theta\epsilon}}(\gamma)$ its best response is to set a quantity. Since the same line of reasoning applies for firm $j$ as well the equilibrium is as characterized in the last two lines of the proposition. ■

Figure 4 displays the different equilibrium region outcomes.

[17] Since the uniqueness of $\hat{\gamma}$ is not necessary for our results, we omit the presentation of this derivative since it is a rather lengthy expression.

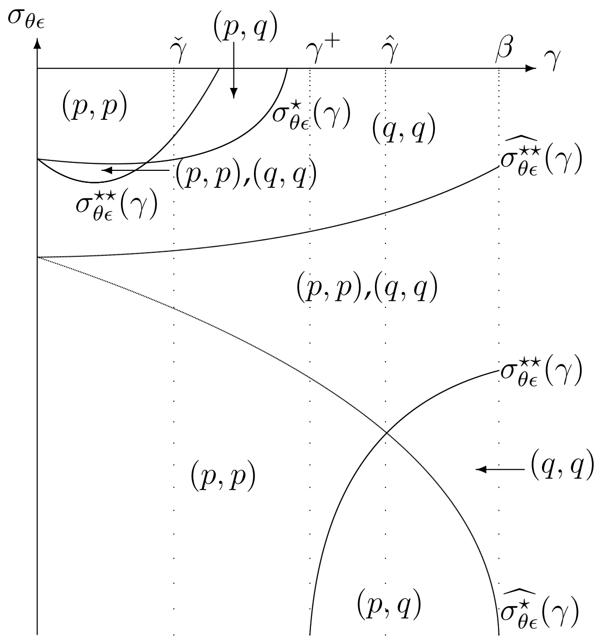


Figure 4: Equilibrium with negative covariance

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