Dynamic Location Games*

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PRELIMINARY AND INCOMPLETE.

Abstract

We study Hotelling location games, in which market entry involves a fixed cost and occurs in an exogenously given sequence. Once a firm has chosen its location, changing it is prohibitively costly. Three questions are of interest. First, how many firms will enter in equilibrium? Second, which locations will they occupy? Third, what is the sequence of settlement? For general distributions of consumers, we determine the equilibrium locations and show that they are unique and independent of the sequence of settlement. Moreover, the sequence of settlement is generically unique. The case with uniformly distributed consumers is non-generic in that respect. It also yields a lower bound on the equilibrium number of firms entering.

JEL-Code: D43, L13, D21

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1 Introduction

Technologies in differentiated product markets are typically characterized by large fixed costs and small or negligible marginal costs. Media markets are one important example since the fixed cost of operation of a TV or radio station or a newspaper are typically large, while the marginal costs of reproduction and serving additional consumers are close to zero.

Moreover, major media such as TV, radio, and newspapers play a key role in modern societies, both with respect to (i) the time individuals spend consuming media products and (ii) the size of the media industry in economic terms.[1] Furthermore, the demand side of media markets is typically composed of various ethnic or socio-demographic groups of consumers, which tend to be highly heterogenous. For the U.S. for example, Waldfogel (2003, p.557) reports that "[R]adio programming formats collectively attracting 60% (46%) of black (Hispanic) listening attract 3% (0.5%) of remaining (chiefly white) listening."[2] Moreover, since many media outlets serve a variety of different geographical markets, they typically face heterogeneity of consumer preferences across these markets. When choosing a location in the product spectrum, a firm thus faces a *multi-market* environment, where a market is defined by ethnic, socio-demographic and geographic characteristics. Since size and significance of each market will typically vary, the dis-

[1] As for (i), according to Anderson and Gabszewicz (2005, Table 3) the average time spent per day by US adults with major media across different age groups ranges from 461 to 514 minutes. Moreover Corneo (2005, Table 1) compares the average annual hours spent for TV and work in 11 Western countries, and finds that the former exceeds the latter in countries such as the US, UK, Germany, Japan, France, Italy and Spain. As for (ii), the main source of income for media companies is revenue from advertising. According to Anderson and Gabszewicz (2005, Table 4), the total advertising volume for the US in 2003 was $245 billion, and thus 2.23% of GDP, which is big business indeed, as noted by Bagwell (2005).

[2] For similar findings for newspapers and magazines, see George and Waldfogel (2003, 2006).

tribution of consumer preferences, aggregated over all these markets, will generally be non-uniform.

We study Hotelling location games in markets with general distributions of consumer preferences. Market entry involves a fixed cost and occurs sequentially. The sequence of entry is exogenously given, and once a firm has chosen its location, changing it is prohibitively costly.[3] Three questions are of particular interest. First, how many firms will enter in equilibrium? Second, which locations will they occupy? Third, what is the sequence of settlement, i.e. which locations are most attractive and will be occupied first? We determine the number of active firms in equilibrium and their locations. These locations are generically unique and independent of the sequence of settlement. Moreover, the sequence of settlement is generically unique.

Our analysis reveals that the special case with uniformly distributed consumers, as analyzed in the previous literature, is quite special indeed. First, among all distribution functions with full support, the uniform yields the smallest number of firms entering in equilibrium. Intuitively when choosing its location in a given interval, a firm is concerned both with attracting as many customers as possible, and with deterring subsequent entry in its neighborhood. For the uniform distribution, the set of entry-deterring locations is a (non-empty) subset of the set of locations that maximize the number of customers, so that any entry-deterring location is always also profit-maximizing. As a result, it is never optimal for a firm to invite further entry when entry deterrence would be feasible. This property does not hold for non-uniform distributions: in general, entry-deterring locations are not necessarily profit-maximizing ones, which tends to increase the number

[3]For example, such cost can be purely physical in the sense of re-locating a plant, or changes in the product itself. In addition, it is typically costly to change an existing brand image of a firm/product in the minds of consumers.

of entrants.

Second, all but at most three firms have identical market shares and earn the same profits in equilibrium. This does not only fit poorly with empirical observations, but it also renders the sequence of settlement indeterminate. As we show, this indeterminacy is knife-edge, however: it disappears once a density with non-zero slope is evoked.

There are several motivations for considering sequential entry. First, it seems a more appropriate description of how entry takes place into real-world (media) markets. Second, following the previous literature, it seems a natural assumption for addressing the issue of entry deterrence. Moreover, as will become clear below, many of our results would continue to hold when firms move simultaneously; all we need is the possibility of further entry in the future. Finally, it is consistent with the robust empirical finding that larger markets attract more firms and more product variety (Waldfogel, 2003).[4]

Throughout, we abstract from price competition. There are three motivations. First, our main interest is in spatial competition, and in determining equilibrium locations. A firm's location choice in product space can be seen as a long-term decision, and in many instances, it is the single most important decision the firm has to make. For example, when comparing, say, the New York Times with the New York Post, the most noticeable feature tends to be their (different) locations in product space, and not differences in prices. Second, and partly as a consequence of the first point, despite its long-term nature, location is the most relevant variable a firm can fully and independently control. This contrasts, e.g., with the revenue a newspaper generates from consumers and ad-

[4]In models with simultaneous entry, the location equilibrium may be in mixed strategies (e.g., for uniformly distributed customers and three firms) so the latter claim may, at most, be true in a stochastic sense. Moreover, no results are known for non-uniform distributions which, as argued above, are empirically more relevant.

vertisers. On the consumer side, there is remarkably little variation in prices over time or across outlets and markets (Lewis, 1995; George and Waldfogel, 2000).[5] Moreover, for many media including newspapers, the most important source of revenue is certainly advertisement.[6] Though advertisement revenue does vary over time, this seems to be driven by the business cycle, rather than a single firm's pricing decisions.[7] Hence, in contrast to location choice, both sources of revenue can be justly seen as, by and large, outside a single firm's sphere of influence. Third, though price competition is certainly relevant in many markets, it can safely be neglected in those where consumer prices are zero to begin with, such as on-air TV and radio broadcasting.[8] Given the significance of these markets, understanding pure location games in these settings is interesting and important in itself.[9]

Building on the seminal work by Hotelling (1929), the issue of entry deterrence in spatial competition was first considered in Prescott and Visscher (1977) who confine attention to the uniform case. Therefore, their model emerges as a special case of ours. Also in a model of sequential entry and with uniformly distributed consumers, Anderson and Engers (2001) analyze endogenous sequencing. Palfrey (1984) considers a class of symmetric non-uniform distributions, where the number of entrants is exogenous. Callander (2005) and Loertscher and Muehlheusser (2006) study models where competition takes

[5] According to the latter, in 2000, 75% of general interest newspapers were sold at 50 cent per copy.

[6] According to George and Waldfogel (2000), about 80% of newspaper revenue is derived from advertisement.

[7] For example, using US data provided on www.tvb.org, one calculates a coefficient of correlation of 0.72 between the change rates of GDP and total advertisement revenue, respectively.

[8] Another question concerns price setting for advertisements. If each media market is small because, say, it is local and if each firm within these markets is small, compared to the national or global market, then these firms can also be assumed to be price takers on the advertisement market.

[9] Politics is another (arguably not less important) example, where voters can typically only choose from a small number of parties and political programs, and where competition is purely spatial.

place in several, heterogenous markets, so that the resulting aggregate distribution of consumer preferences is non-uniform. Again, attention is confined to a special class of distributions and to an exogenous number of firms, respectively.

If one assumes that the firms in our model are price takers, they can also be interpreted as platforms in a two-sided market, where one side of the market consists of consumers and the other one of advertisers. Consequently, our paper is also related to the recent literature on two-sided media markets such as Gabszewicz, Laussel, and Sonnac (2001), Anderson and Gabszewicz (2005) or Ambrus and Reisinger (2006).

The remainder of the paper is organized as follows. Section 2 introduces the model. In Section 3 we study in turn locations that are optimal absent further entry and locations that deter further entry. Section 4 contains preliminary results on equilibrium properties. In Section 5 we derive the equilibrium locations for quasiconcave and non-quasiconcave densities, respectively. The equilibrium sequences of settlement are addressed in Section 6. Section 7 discusses our results and compares them with previous findings. Section 8 concludes.

2 The model

Consider a product market with a unit mass of consumers distributed along the $[0,1]$-interval according to the cumulative distribution function $F(\cdot)$, which is continuous and differentiable almost everywhere. Its density $f(\cdot)$ satisfies $f(y) > 0$ for all $y \in [0,1]$ where it exists. Each consumer patronizes the closest firm.

There are N firms who can potentially enter the market, where N is a large but finite number.[10] Firms move sequentially and the sequence of moves is exogenously given. If

[10]Finiteness of N matters only for the uniqueness of equilibrium configurations; see Section 7 below.

firm i is given the move, it decides whether or not to enter the market. If it enters, it incurs a fixed cost $T > 0$ and chooses a location $x_i \in [0, 1]$. In either case, its decision is observed by all firms moving subsequently. The profit of each active firm, gross of the entry cost T, is equal to mass of consumers it attracts. To rule out trivial cases, we assume $T < \frac{1}{2}$ such that the market can at least support two firms. Apart from the possibility that later entrants face a less attractive choice set, no costs are associated with entering later.[11] Finally, for convenience we assume that firms stay out when indifferent.

3 Concepts

In this section, we develop several concepts which are crucial for the following equilibrium analysis. We begin with the optimal location of a firm in a given interval under the assumption of no subsequent entry, and then turn to the issue of entry deterrence.

Throughout the paper, we refer to an interval (L, R) as one where the points L and R are already **occupied** by competitors, and which is **empty** in the sense that no firm is located in its interior.

3.1 Optimal locations absent further entry

Consider a firm entering in an interval (L, R).[12] Under the assumption of *no further subsequent entry* in this interval, when entering at location $x \in (L, R)$, the firm's profit is

$$\pi(x, L, R) := F\left(\frac{x+R}{2}\right) - F\left(\frac{x+L}{2}\right).$$

[11] See Section 8 for a discussion.

[12] Note that while the interval (L, R) is open by definition, firms are not a priori prohibited to choose identical locations; as will be shown below, however, such behavior is inconsistent with equilibrium.

That is, it attracts all customers between the midpoints between its own location and the locations of its competitors to the right and left, respectively. Note that the "reach" of the firm's customer base, denoted by $\Delta(L,R)$, is simply half the interval length, and thus independent of x:

$$\Delta(L,R) := \frac{R-L}{2}.$$

Lemma 1 *For any location $x \in (L,R)$,*

$$\frac{\partial \pi(x,L,R)}{\partial L} = -\frac{1}{2} f\left(\frac{x+L}{2}\right) < 0 \quad \textit{and} \quad \frac{\partial \pi(x,L,R)}{\partial R} = \frac{1}{2} f\left(\frac{x+R}{2}\right) > 0.$$

The lemma is obvious and requires no proof. In words it says that the profit of a firm, whose location is fixed at some $x \in (L,R)$, is strictly increasing in the distance to its closest competitors.

Some additional notation is useful:

Definition 1 *Denote by*

(i) $X^*(L,R)$ *the set of optimal locations in the interval* (L,R)*:*

$$X^*(L,R) := \arg\max_{x\in(L,R)} \pi(x,L,R).$$

An element of this set is denoted by $x^*(L,R)$.[13]

(ii) $\pi^*(L,R)$ *the firm's profit when locating optimally:*

$$\pi^*(L,R) := \pi(x^*,L,R) \quad \textit{for} \quad x^* \in X^*(L,R).$$

(iii) $\hat{X}(\underline{z},\overline{z},L,R)$ *the set of optimal locations in the interval* (L,R) *when the choice set is restricted to some interval* $[\underline{z},\overline{z}] \subset (L,R)$*:*

$$\hat{X}(\underline{z},\overline{z},L,R) := \arg\max_{x\in[\underline{z},\overline{z}]} \pi(x,L,R)$$

[13] For example, when $f(\cdot)$ is uniform, $X^*(L,R) = (L,R)$ holds; see Section 7 below.

An element of this set is denoted by $\hat{x}(\underline{z}, \overline{z}, L, R)$

(iv) $\hat{\pi}(\underline{z}, \overline{z}, L, R)$ *the firm's profit when choosing one of these optimal (restricted) locations:*

$$\hat{\pi}(\underline{z}, \overline{z}, L, R) = \pi\left(\hat{x}, L, R\right) \quad \textit{for} \quad \hat{x} \in \hat{X}(\underline{z}, \overline{z}, L, R).$$

(v) $L^+ := L + \epsilon$ *and* $R^- := R - \epsilon$ *for arbitrarily small* $\epsilon > 0$*, the smallest and the largest possible locations in* (L, R)*, respectively.*[14]

Lemma 2 $\pi^*(L, R)$ *and* $\hat{\pi}(\underline{z}, \overline{z}, L, R)$ *strictly decrease in* L *and strictly increases in* R*.*

Proof. In the proof, we confine attention to π^* as the arguments regarding $\hat{\pi}$ are completely analogous. The proof for the reaction of $\pi^*(L, R)$ to changes in L and R relies on a revealed preference argument: Fix some $x^*(L, R) \in X^*(L, R)$ and suppose that the competitor to the left moves to some $L' > L$. We have to consider two cases.

Case 1: $x^*(L, R) \in X^*(L', R)$. From Lemma 1, it follows directly that $\pi(x^*(L, R), L', R) < \pi^*(L, R)$.

Case 2: $x^*(L, R) \notin X^*(L', R)$. To see that $\pi^*(L', R) < \pi^*(L, R)$ holds, suppose otherwise that $\pi^*(L', R) \geq \pi^*(L, R)$. By definition of $x^*(L, R)$, however, $\pi^*(L, R) \geq \pi(x^*(L', R), L, R)$. Therefore, if the first inequality holds, then so does

$$\pi^*(L', R) \geq \pi(x^*(L', R), L, R).$$

But this is a contradiction to Lemma 1. Completely analogous arguments apply to changes of R. ■

[14]Of course, since the interval (L, R) is open, L^+ and R^- are not well-defined in a continuous framework. We follow the standard notion in the literature where the continuous case emerges as the limit of a discrete choice set with "grid size" ϵ, where $\epsilon \to 0$.

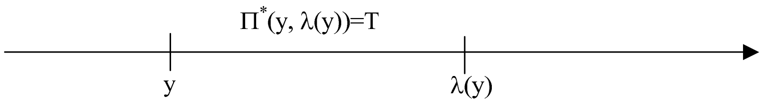

Figure 1: y and $\lambda(y)$.

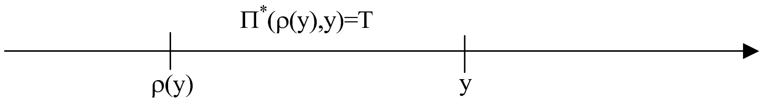

Figure 2: $\rho(y)$ and y.

3.2 Entry-deterring locations

The following concepts are useful for addressing the issue of entry deterrence and for determining equilibrium configurations. Thereby a distinction has to be made between (i) entry deterrence with respect to an already occupied location, and (ii) with respect to one of the (unoccupied) boundary points $\{0,1\}$ of the product spectrum.

Definition 2 *(i) Define $\lambda(y)$ and $\rho(y)$ such that*

$$\pi^*(y, \lambda(y)) = T \quad \textit{and} \quad \pi^*(\rho(y), y) = T,$$

for any occupied locations $y \in [0, \rho(1)]$ and $y \in [\lambda(0), 1]$, respectively.

(ii) Let

$$\lambda^0 := F^{-1}(T) \qquad \textit{and} \qquad \rho^1 := F^{-1}(1-T).$$

Note first that although the notation does not make this explicit, it is clear that $\lambda(\cdot)$ and $\rho(\cdot)$ also depend on T. As for part (i), consider Figures 1 and 2 for illustrations and note that $\lambda(y) > y$ and $\rho(y) < y$. Observe also that $\lambda(\cdot)$ is the inverse of $\rho(\cdot)$, i.e. $\rho(\lambda(y)) = y = \lambda(\rho(y))$.[15] Intuitively, with competitors located at y and $\lambda(y)$, an entrant

[15]Thus, $\lambda(y) \leq 1$ holds for all $y \leq \rho(1)$ and $\rho(y) \geq 0$ holds for all $y \geq \lambda(0)$.

would get exactly T when locating optimally in the interval $(y, \lambda(y))$ and, consequently, prefers not to enter. The intuition for $\rho(y)$ is analogous.

Because $\partial\pi(x, L, R)/\partial L < 0$ and $\partial\pi(x, L, R)/\partial R > 0$ for any $x \in (L, R)$, $\lambda(\cdot)$ and $\rho(\cdot)$ are unique. As will be shown below, for any occupied location y, $\lambda(y)$ is therefore the largest entry-deterring location to the right of y. Analogously, $\rho(y)$ is the smallest entry-deterring location to the left of y.

Part (ii) is an appropriate adaption of these definitions to unoccupied boundary points: If the left boundary point is not occupied while a firm is located at λ^0, an entrant would just be deterred from entering in the interval $[0, \lambda^0)$.[16] An analogous argument applies to the right boundary. Note that our previous assumption $T < \frac{1}{2}$ implies $\lambda^0 < \rho^1$.[17]

Lemma 3 *(i) $\lambda(y)$ and $\rho(y)$ strictly increase in y.*

(ii) For any two occupied locations L, R with $L < R$,

$$\lambda(L) < R \Leftrightarrow \rho(R) > L \Leftrightarrow \pi^*(L, R) < T.$$

(iii) For any occupied locations $y \in [0, \rho(1)]$ and $y \in [\lambda(0), 1]$, respectively,

$$T < F(\lambda(y)) - F(y) \leq 2T \quad \text{and} \quad T < F(y) - F(\rho(y)) \leq 2T.$$

[16] Note that an entrant's optimal location in this case would be λ^{0-} for any distribution, since he gets the whole hinterland. Moreover, we have $\lambda^0 < \lambda(0)$, because the only difference refers to whether or not the end point 0 is occupied. By definition of λ^0, this is not the case, and so a firm locating at some $x \leq \lambda^0$ gets the whole hinterland to the left of x. However, when the end point 0 is occupied as is the case by definition of $\lambda(0)$, this hinterland is shared with the firm at 0. Analogous reasoning establishes that $\rho(1) < \rho^1$.

[17] It is now clear why the case $T > \frac{1}{2}$ is trivial: either $\frac{1}{2} < T < 1$ (i.e. $0 < \rho^1 \leq \lambda^0 < 1$), so that the first firm would optimally enter somewhere in the interval $[\rho^1, \lambda^0]$ thereby forestalling further entry. Or $T > 1$ (i.e. $\rho^1 \leq 0 < 1 \leq \lambda^0$), in which case the market could not even support one firm.

(iv) For y given, $\partial\lambda/\partial T > 0$ and $\partial\rho/\partial T < 0$.

Proof. Part (i) Suppose, for notational simplicity, that optimal locations are unique. By definition, when locating at $x^*(y, \lambda(y))$, an entrant gets T. When the firm to the left is instead located at some $y' > y$, $\pi^*(y', \lambda(y)) < T$ follows from Lemma 2. This Lemma also implies that $\pi^*(y', \lambda(y')) = T$ can hold only if $\lambda(y') > \lambda(y)$. A completely analogous argument establishes that $\rho(\cdot)$ is also increasing.

Part (ii) By construction $\lambda(\rho(R)) = R$ and by part (i), $\lambda' > 0$. Hence, $\rho(R) < L$ implies $\lambda(L) > R$. That this implies $\pi^*(L, R) < T$ follows from Definition 2.

Part (iii) $F(\lambda(y)) - F(y) > T$ and $F(y) - F(\rho(y)) > T$ follows trivially from the definition of $\lambda(\cdot)$ and $\rho(\cdot)$. The remainder of the proof for the statement with respect to $\lambda(y)$ relies on the validity of the following claim:

Claim: $F(\frac{\lambda(y)+y}{2}) - F(y) \leq T$ and $F(\lambda(y)) - F(\frac{\lambda(y)+y}{2}) \leq T$

Suppose otherwise that $F(\frac{\lambda(y)+y}{2}) - F(y) > T$. Then an entrant could locate at y^+ (i.e. infinitesimally close to the right of y) and get $\pi(y^+, y, \lambda(y)) = F(\frac{\lambda(y)+y}{2}) - F(y) > T$ which contradicts the definition of $\lambda(\cdot)$. An analogous argument establishes the second part of the claim. The proof for the statement with respect to $\rho(y)$ is completely analogous.

Part (iv) By definition, when locating at $x^*(y, \lambda(y))$, an entrant gets T. When T increases to $T' > T$, the set of optimal locations does not change, and thus $\pi^*(y, \lambda(y)) < T'$ holds. Thus, by Lemma 2, for a given y, $\pi^*(y, \lambda(y)) = T'$ can hold only if $\lambda(y)$ increases. A completely analogous argument establishes that $\partial\rho/\partial T < 0$. ■

Note that part (ii) of the Lemma rules out "asymmetric" cases such as $\rho(R) < L < \lambda(L) < R$ and $L < \rho(R) < R < \lambda(L)$.

4 Preliminaries: Equilibrium properties

The remainder of the paper is devoted to characterizing subgame perfect equilibria of dynamic location games, which from now on we simply refer to as "equilibria". We first derive some general equilibrium properties, and start with an implication of Lemma 3, which will be used frequently throughout:

Corollary 1 *Three occupied locations L, x, R, where $L < x < R$ are inconsistent with equilibrium if*

$$\rho(R) \leq L < x < R \leq \lambda(L). \tag{1}$$

When (1) holds, then $\pi(x, L, R) \leq T$ for all $x \in (L, R)$ follows from Lemma 3. So the firm at location x could profitably deviate by staying out of the market.

4.1 Number of entrants in a given interval

Denote by # the number of firms entering in equilibrium in a given interval (L, R).

Theorem 1 *In any equilibrium,*

(i) $\# = 0$ *if* $\rho(R) \leq L < R \leq \lambda(L)$

(ii) $\# \in \{1, 2\}$ *if* $L < \rho(R) < \lambda(L) < R$

(iii) $\# \geq 2$ *if* $L < \lambda(L) < \rho(R) < R.$

Proof. Recall first that because of the symmetry of $\lambda(\cdot)$ and $\rho(\cdot)$, the cases $\rho(R) < L < \lambda(L) < R$ and $L < \rho(R) < R < \lambda(L)$ cannot occur. So only the three cases stated in the theorem need to be considered.

Part (i) By definition of $\lambda(L)$ and $\rho(R)$, and from Corollary 1, profitable entry in the interval (L, R) is not possible in this case, and thus no firm will enter in equilibrium.

Part (ii) Label subsequent entrants by $i, i+1, i+2, \ldots$. We show that a) at most two firms enter in equilibrium, and b) at least one enters.

a) At most two firms enter

If the first entrant i enters at some $x_i \in [\rho(R), \lambda(L)]$, then by definition of $\lambda(\cdot)$ and $\rho(\cdot)$, there will be no further entry in this interval. So consider the case where $x_i \notin [\rho(R), \lambda(L)]$, and suppose $x_i \in (L, \rho(R))$. The case $x_i \in (\lambda(L), R)$ is completely analogous and thus omitted. By Corollary 1, if subsequently $i+1$ enters, it must enter at some $x_{i+1} > \lambda(L)$: For $x_{i+1} \in (L, x_i]$, firm $i+1$ itself would incur a loss, for $x_{i+1} \in (x_i, \lambda(L)]$, firm i would do so. For two firms to enter, it therefore has to be the case that one, say i,locates at $x_i < \rho(R)$ and the other one at $x_{i+1} > \lambda(L)$. But now a third firm cannot profitably enter because at least one of the firms would not break even. This follows again from Corollary 1: For $x_{i+2} \in (L, x_i)$ or $x_{i+2} \in (x_{i+1}, R)$, firm $i+2$ does not break even, for $x_{i+2} \in (L, \lambda(L))$, i does not break even, and for $x_{i+2} \in (\lambda(L), x_{i+2})$, firm $i+1$ does not break even.

b) At least one firm enters

Three cases have to be considered:

Case 1: There is a $x^*(L, R) \in [\rho(R), \lambda(L)]$. In this case, the first entrant chooses this location, thereby preventing further entry. Moreover $\pi^*(L, R) > T$ since $L < \rho(R) < \lambda(L) < R$.

Case 2: There is no $x^*(L, R) \in [\rho(R), \lambda(L)]$ but $\hat{\pi}(\rho(R), \lambda(L), L, R) > T$. In this case, at least one firm will enter since $\hat{x}(\rho(R), \lambda(L), L, R)$ is a profitable and entry-deterring location. Whether one or two firms enter depends on whether the first firm i prefers an alternative location, thereby inducing subsequent entry, to $\hat{x}(\rho(R), \lambda(L), L, R)$ and thereby deterring entry.

Case 3: $\hat{\pi}(\rho(R), \lambda(L), L, R) \leq T$. Observe first that this implies $x^*(L, \lambda(L)) < \rho(R)$ and $x^*(\rho(R), R) > \lambda(L)$. We need to show that at least one firm enters, assuming equilibrium behavior by firms moving subsequently. That is, we have to show that there exists some $x_i \in (L, R)$ such that i's profit at x_i exceeds T if all subsequent firms play optimally. Let i occupy the location $x^*(L, \lambda(L))$. Observe first that there will be no subsequent entry to the left of firm i, because by Corollary 1, for any location $y \in (L, x_i)$, $\pi(y, L, x_i) < T$ holds. A necessary condition for i not to break even at $x^*(L, \lambda(L))$ is therefore that (at least) one other firm, say, $i+1$ enters to its right at some $x_{i+1} \leq \lambda(L)$. Only in this situation will i be "trapped" inside the $[L, \lambda(L)]$ interval (Corollary 1). So assume $x_{i+1} \leq \lambda(L)$. But for $i+1$ to enter at x_{i+1} in equilibrium, it must be the case that $i+1$ earns more than T either by deterring further entry or by "pushing" any subsequent entrant far enough to the right. But if $i+1$ earns more than T at x_{i+1} with $x_i > L$ to its left, then i could have chosen the location x_{i+1} itself, whereby it would have earned strictly more than $i+1$ now does. Therefore, i can guarantee itself a profit that is larger than T. Consequently, at least on firm will enter in equilibrium.

Part (iii) The proof relies on the validity of the following claim:

Claim: At least one firm can profitably enter either in the interval $(L, \lambda(L))$ or in the interval $(\rho(R), R)$.

We prove the claim for the case where the first entrant i enters in the interval $(L, \lambda(L))$, for the other one it is completely analogous. Suppose the first entrant i locates at $x_i = x^*(L, \lambda(L))$. Since $x^*(L, \lambda(L)) < \lambda(L)$, there will be no more entry to the left of x_i (by Corollary 1). Suppose without loss of generality that the closest firm to the right of firm i will be firm $i+1$ at some location x^0_{i+1}: If $x^0_{i+1} > \lambda(L)$, then $\pi(x^*(L, \lambda(L)), L, x^0_{i+1}) > T$.

Thus, as above, the critical case is $x^0_{i+1} \leq \lambda(L)$ such that firm i would not break even (again by Corollary 1). Note that firm $i+1$ would choose such a position only if $\pi(x^0_{i+1}, x_i, z) > T$ where $z > \lambda(x_i)$ is the closest firm to the right of firm $i+1$. But then, firm i could itself locate at $x_i = x^0_{i+1}$ and earn $\pi(x^0_{i+1}, L, z) > T$ since there will be no further entry in the interval (L, x_{i+1}). Consequently, there always exists a location in the interval $(L, \lambda(L))$ such that entry is profitable for at least one firm.

How many more firms enter depends on the location of $\lambda(x_i)$. If $\lambda(x_i) > \rho(R)$, we are in part (ii), where it was shown that at least one more firm enters. If $\lambda(x_i) < \rho(R)$, then we are again in part (iii) in which case at least two more firms enter. ■

Intuitively, in part (i) the market size in a given interval is too small to support profitable entry. As the market size increases, or equivalently, if the fixed cost decreases, so that we are in the case described by part (ii), at least one entrant can profitably enter in the interval. Whether this first entrant optimally forestalls further entry or invites entry by one more firm depends on the distribution of consumers. As the market size increases even further, so that we are in the case described by part (iii), the first entrant can no longer deter further entry, so that in this case at least two firms enter.

4.2 Distance between neighboring firms

Apart from the number of firms entering in equilibrium in a given interval, we can also say something about distances between firms in any equilibrium. We refer to two firms at locations L and R as **neighbors** when the interior of the interval (L, R) is empty. We start with the following corollary of Theorem 1:

Corollary 2 *In any equilibrium, two firms at locations L and R, respectively, where $L < R$ are neighbors if and only if $\pi^*(L, R) \leq T$.*

Part (i) of Theorem 1 implies that there will be no further entry if $\pi^*(L, R) \leq T$, which, as will be recalled, is equivalent to $\rho(R) \leq L \Leftrightarrow R \leq \lambda(L)$. To see that there is entry if $\pi^*(L, R) > T$, observe first that $\pi^*(L, R) > T \Leftrightarrow \rho(R) > L \Leftrightarrow R > \lambda(L)$, for which case(s) parts (ii) and (iii) of Theorem 1 say there will be entry. Hence, L and R cannot be neighbors if $\pi^*(L, R) > T$.

Theorem 2 *For any three neighboring equilibrium locations L, x, R satisfying $L < x < R$, the following condition must hold:*

$$\rho(x) \leq L < \rho(R) \leq x \leq \lambda(L) < R \leq \lambda(x). \tag{2}$$

Proof. From Corollary 1, if $\rho(R) \leq L$, or if $R \leq \lambda(L)$, the firm at x could profitably deviate by staying out. Moreover from Corollary 2, when the distance between the firm at x and its neighbors exceeds $x - \rho(x)$ and $\lambda(x) - x$, respectively, then there will be entry in-between, contradicting that x and L (resp. x and R) are neighbors. ■

Intuitively, the Theorem is a statement about distances between neighboring firms in any equilibrium: First, the *minimum distance* between the firms located at x and L must be strictly larger than $x - \rho(R)$. Otherwise when $L > \rho(R)$ (i.e. when L is "too close"), then by Corollary 1, the firm at x does not break even. The same argument applies to the right-hand side when $R < \lambda(L)$.

Second, the *maximum distance* between the firm located at x and its neighbors at L and R, is $x - \rho(x)$ and $\lambda(x) - x$, respectively. Otherwise, by Corollary 2, there would be entry in-between, contradicting that the firms at locations x and L (resp. x and R) are neighbors.

Although we have not a priori ruled out the possibility that firms choose identical locations, the following implication of Theorem 2 establishes that this will not happen in equilibrium:

Corollary 3 *In any equilibrium, any location $x \in [0,1]$ is occupied by at most one firm.*

Proof.

As shown in Theorem 2, in any equilibrium the maximum distance between a firm at location x and its neighbors to the left and right is $x - \rho(x)$ and $\lambda(x) - x$, respectively.

Moreover, as shown in the proof of part (iii) of Lemma 3, $F(\frac{\lambda(x)+x}{2}) - F(x) \leq T$ and $F(x) - F(\frac{\rho(x)+x}{2}) \leq T$, so that for the *total* profit generated at location x, $F(\frac{\lambda(x)+x}{2}) - F(\frac{\rho(x)+x}{2}) \leq 2T$ holds. Therefore, independent of how this profit is shared between the firms located at x, at most one can break even. ■

4.3 Range of product variety

Theorem 3 *In any equilibrium, the left- and rightmost occupied locations will be λ^0 and ρ^1, respectively.*

Proof. Assume the smallest location occupied in equilibrium is some $x > \lambda^0$. Then a firm can profitably enter to the left of x, which is a contradiction. On the other hand, assume the smallest location is $x < \lambda^0$. Depending on the location of the closest neighbor to the right of the firm at x, denoted as y, two cases can occur. If $y \leq \lambda^0$, the firm at x does not break even, contradicting that this is an equilibrium location. If $y > \lambda^0$, the profit of the firm at x would have been larger had it located further to the right. Thus, $x < \lambda^0$ cannot be an equilibrium location. Completely analogous reasoning applies to the other boundary point. ■

Thus, the result is consistent with intuitive notion that when only few consumers are located near the boundaries of the product spectrum, firms will cater these boundaries less, which results in a smaller range of product variety.

5 Equilibrium locations

We focus first on the locations that are occupied in equilibrium. We start with the class of distribution functions with quasiconcave densities and then turn to non-quasiconcave densities.[18]

5.1 Quasiconcave densities

Preliminaries To determine the equilibrium locations for the case of quasiconcave densities, we need the following preliminary results:

Lemma 4 *(Monotonicity) Let f be quasiconcave over an interval (L, R) and consider any $x^*(L, R)$ and any locations y, z.*

(i) If $x^*(L, R) < y < z \leq R$, then $\pi(y, L, R) \geq \pi(z, L, R)$ holds.

(ii) If $L \leq y < z < x^*(L, R)$, then $\pi(y, L, R) \leq \pi(z, L, R)$ holds.

Proof: Consider without loss of generality part (i). If f is monotone decreasing over the interval (L, R), then $x^*(L, R)$ is unique and equal to L^+: Recall that when locating in (L, R), the reach of a firm's customer base is $\Delta(L, R) = \frac{R-L}{2}$, and thus independent of the firm's location. However, the firm's profit strictly increases as it moves closer to x^* since the density is strictly higher to the left.

If f is hump-shaped, then x^* is interior, satisfying $f(\frac{x^*+L}{2}) = f(\frac{x^*+R}{2})$. Consequently, for any $x > x^*$, $f(\frac{x+L}{2}) \geq f(\frac{x+R}{2})$ holds. Thus, when moving left from x the marginal

[18] Let A be a convex set. Then, a function $f : A \to \mathbb{R}$ is called quasiconcave if and only if $f(x) \geq t$ and $f(x') \geq t$ for any $x, x' \in A$ implies $f(\alpha x + (1-\alpha)x') \geq t$ for any $\alpha \in [0, 1]$. Obviously, any (weakly) monotone function is quasiconcave. Also any function $f(x)$ for $x \in [0, 1]$ with $f'(0) \geq 0$ and where f' changes its sign only once is quasiconcave. However, a function for which the sign of f' changes twice (or only once, but where $f'(0) < 0$) is not quasiconcave.

gain on the left will be (weakly) larger than the marginal loss on the right. Hence, $\pi(y, L, R) \geq \pi(z, L, R)$ for any $z > y > x^*$ follows. ■

Note that the inequality statements in the Lemma become strict when f is either monotone, or when $f' = 0$ for at most one location.

Lemma 5 *Assume $x^*(L, R)$ is interior, satisfying $f\left(\frac{x^*+L}{2}\right) = f\left(\frac{x^*+R}{2}\right)$ and $f'\left(\frac{x^*+L}{2}\right) > 0 > f'\left(\frac{x^*+R}{2}\right)$. Then:*

$$\frac{\partial x^*}{\partial L} < 0 \quad \text{and} \quad \frac{\partial x^*}{\partial R} < 0.$$

Proof. Take the total differential of the first order condition $f\left(\frac{x^*+L}{2}\right) = f\left(\frac{x^*+R}{2}\right)$ with respect to L, and R, respectively: As for L, we have

$$\frac{1}{2} f\left(\frac{x^* + L}{2}\right)\left(\frac{\partial x^*}{\partial L} + 1\right) = \frac{1}{2} f\left(\frac{x^* + R}{2}\right)\frac{\partial x^*}{\partial L}$$

which is equivalent to

$$\frac{\partial x^*}{\partial L} = \frac{-f\left(\frac{x^*+L}{2}\right)}{f\left(\frac{x^*+L}{2}\right) - f\left(\frac{x^*+R}{2}\right)} < 0$$

since $f'\left(\frac{x^*+L}{2}\right) > 0 > f'\left(\frac{x^*+R}{2}\right)$. Analogously, for the total differential with respect to to R we get

$$\frac{1}{2} f\left(\frac{x^* + L}{2}\right)\frac{\partial x^*}{\partial R} = \frac{1}{2} f\left(\frac{x^* + R}{2}\right)\left(\frac{\partial x^*}{\partial R} + 1\right)$$

which is equivalent to

$$\frac{\partial x^*}{\partial R} = \frac{f\left(\frac{x^*+R}{2}\right)}{f\left(\frac{x^*+L}{2}\right) - f\left(\frac{x^*+R}{2}\right)} < 0.$$

■

Consider now an interval (L, R) satisfying $L < \rho(R) < \lambda(L) < R$. From Theorem 1 we know that at least one, and at most two firm(s) will enter in this interval. The next result establishes that when f is quasiconcave, exactly one firm will enter:

Lemma 6 *Let f be quasiconcave over an interval (L, R) satisfying $L < \rho(R) < \lambda(L) < R$, exactly one additional firm will enter. This firm locates at*

(i) $\lambda(L)$ *if* $f' > 0$,

(ii) $\rho(R)$ *if* $f' < 0$,

(iii) $\hat{x}(\rho(R), \lambda(L), L, R)$ *if f is hump-shaped.*

Proof. That at least one additional firm enters follows from Theorem 1. So we are left to show that further entry deterrence is optimal for the first entrant. There are two cases: Either f is monotone as in Figures 3 or 4, or it is hump shaped as in Figure 5. We first consider the monotone case, focusing without loss of generality on increasing functions.

Let i be this entrant and suppose to the contrary that i accommodates further entry either by choosing $x_i \in (\lambda(L), R)$ or $x_i \in (L, \rho(R))$. If $x_i \in (L, \rho(R))$ is an equilibrium outcome, then the next entrant will deter entry (by Theorem 1). He optimally does so by choosing $\hat{x}(\rho(x_i), \lambda(L), L, x_i) = \lambda(L)$. But in this case, x_i is in between two neighbors who are in the interval $[\rho(R), R]$. By Corollary 1, i cannot break even. Hence, this cannot be optimal for i.

If, on the other hand, i chooses $x_i \in (L, \rho(R))$, the subsequently entering firm will, again, deter entry. It optimally does so by choosing $\hat{x}(\rho(R), \lambda(x_i), x_i, R) = \lambda(x_i)$. In this case, i earns strictly less than he would had he located at $\lambda(L)$ and thereby deterred entry: Both the length of the interval he captures is now $\frac{\lambda(x_i)-L}{2}$ instead of $\frac{R-L}{2}$ which he would cover when deterring entry, *and* the density over this interval is smaller than the density he would get when deterring entry. Hence, i will optimally deter entry. He optimally does so by locating at $\lambda(L)$. Note that i's profit will also be strictly higher than T, since $x^*(L, \lambda(L)) = \lambda(L)^-$ so that $\pi^*(L, \lambda(L)) = T$ and $R > \lambda(L)$ (by Lemma 2).

Consider now the case where f is hump-shaped. If there exist $x^*(L, R) \in [\rho(R), \lambda(L)]$ the claim is established since these optimal locations in (L, R) is themselves entry-deterring. So assume there is no $x^*(L, R) \in [\rho(R), \lambda(L)]$, and without loss of generality consider some $x^*(L, R) < \rho(R)$ (the case $x^*(L, R) > \lambda(L)$ is completely analogous). By Lemma 4, $\hat{x}(\rho(R), \lambda(L), L, R) = \rho(R)$ and $\hat{\pi}(\rho(R), \lambda(L), L, R) > \pi(x, L, R)$ for any $x > \lambda(L)$. Hence, the optimal entry deterring location yields a larger profit than locating to the right of $\lambda(L)$. Hence, i will never choose $x_i \in [\lambda(L), L)$. So assume $x_i \in (L, \rho(R))$. Then, either (a) the subsequent (and, by Theorem 1, last) entrant's unconstrained optimal location in (x_i, R), $x^*(x_i, R)$, will be given by the first order condition. In this case, though, by Lemma 5, $x^*(x_i, R) < x^*(L, R)$ and by Lemma 4, the last entrant's optimal entry deterring location will be $\rho(R)$. Or, (b) $x^*(x_i, R)$ will be a corner solution, in which case it will be x_i^+. Hence, again, the last entrant's optimal entry deterring location is $\rho(R)$. Both in case (a) and (b), though, x_i will be "trapped" inside $[L, \lambda(L)]$, where it cannot break even due to Corollary 1.

Observe also that by deterring entry, i nets a profit that is strictly larger than T. This is obvious for $x^*(L, R) \in [\rho(R), \lambda(L)]$. If such locations do not exist, then, by Lemma 4, for any $x^*(L, R) \in (L, \rho(R))$, we have $\hat{x}(\rho(R), \lambda(L), L, R) = \rho(R)$. Hence, $\pi(\rho(R)^+, \rho(R), R) = T$, but $\pi(\rho(R), L, R) > \pi(\rho(R)^+, \rho(R), R) = T$. Similarly, for any $x^*(L, R) > \lambda(L)$, $\hat{x}(\rho(R), \lambda(L), L, R) = \lambda(L)$ and consequently $\pi(\lambda(L), L, R) > T$. ■

The result also illustrates again the potential difference between optimal locations in a given interval *absent* further entry, $^*(\cdot)$, and optimal entry-deterring ones, $\hat{x}(\cdot)$: For example when $f' > 0$, both are unique and we have $x^*(L, R) = R^-$ which, however,

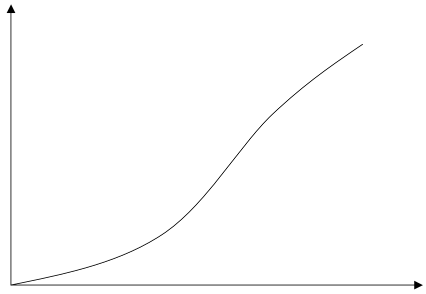

Figure 3: $f' > 0$

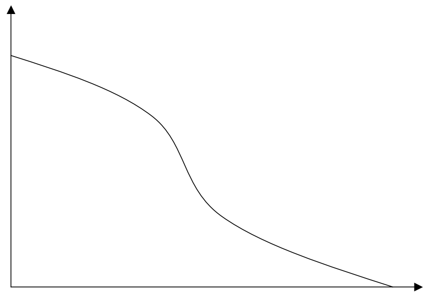

Figure 4: $f' < 0$

would invite further entry. As a result, the first entrant optimally chooses the best of all entry-deterring locations, $\hat{x}(\rho(r), \lambda(L), L, R) = \lambda(L)$, thereby earning profit $\hat{\pi}(\cdot) < \pi^*(\cdot)$.

Monotone densities Let us now look at the cases $f'(x) > 0$ and $f'(x) < 0$, respectively, for all $x \in [0, 1]$ and start with the former.[19]

For λ^0 and ρ^1 as defined in part (ii) of Definition 2, let

$$\lambda^1 \equiv \lambda(\lambda^0), \quad \lambda^2 \equiv \lambda(\lambda^1), \quad \ldots \quad , \lambda^{k+1} \equiv \lambda(\lambda^k),$$

and

$$\rho^2 \equiv \rho(\rho^1), \quad \rho^3 \equiv \rho(\rho^2), \quad \ldots \quad , \rho^{j+1} \equiv \rho(\rho^j).$$

[19] The restriction to strictly increasing or strictly decreasing functions is almost without loss of generality. The analysis goes through if f is flat on some parts of the interval [0,1] as long as $f'(x) > 0$ occurs sufficiently often. More precisely, a sufficient condition for the analysis to go through is that, in the case of increasing functions, in any interval $[\lambda^k, \lambda^{k+1}]$ (defined below) there is an x such that $f'(x) > 0$. This will make sure that the best responses are unique.

Let n be the largest integer such that

$$\lambda^n < \rho^1.$$

That such an n exists and is unique follows from the monotonicity of $\lambda(\cdot)$.

Theorem 4 *Assume $f'(y) > 0$ for all $y \in [0, 1]$. Then, the set of locations occupied in any equilibrium is unique. It is*

$$\{\lambda^0, \lambda^1, .., \lambda^n, \rho^1\},$$

and the number of firms who enter in equilibrium is $n + 2$.

Proof: *Existence.* We first show that $\{\lambda^0, \lambda^1, .., \lambda^n, \rho^1\}$ are equilibrium locations. We know from Theorem 3 that λ^0 and ρ^1 are occupied in any equilibrium. Thus, we are left to show that there is an equilibrium outcome where the locations $\{\lambda^1, .., \lambda^n\}$ are occupied and no others (except for λ^0 and ρ^1).

To that end, assume that all firms play the strategy: "Enter to the right of some location x_i only if its closest righthand neighbor, $i+1$, is at some $x_{i+1} > \lambda(x_i)$ and when $\lambda(x_i) < \rho(1)$. If you enter to the right of x_i, enter at $\lambda(x_i)$." Call this the λ-packman strategy.

To see that these strategies are mutual best responses, notice first that not to enter in $[x_i, x_{i+1}]$ if $x_{i+1} \leq \lambda(x_i)$ is, obviously, a best response. Second, if x_{i+1} is the future righthand neighbor of the entering firm and if $x_{i+1} > \lambda(x_i)$, then entry at $\lambda(x_i)$ is optimal within the interval $[x_i, x_{i+1}]$, as we know from Lemma 6. (And since all better options are taken before by other firms in case there are better options, at some point some firm will enter here.) If the righthand neighbor x_{i+1} has not taken its location yet but plays the λ-packman strategy, then a fortiori $\lambda(x_i)$ is optimal for the entrant: Not only is it the largest location that deters entry to its left, but it will also push its righthand neighbor $i+1$ as far to the right as possible. Thus, λ-packman is a best response to itself, whence it follows that $\{\lambda^1, .., \lambda^n\}$ are the only locations other than λ^0 and ρ^1 that are occupied in equilibrium if all play a λ-packman strategy.

Uniqueness. If the future left-hand and righthand neighbor to some entrant are given at, again, x_i and x_{i+1}, respectively, then $\lambda(x_i)$ is still the best response of the entrant in $[x_i, x_{i+1}]$. Observe also that it is the unique best response. So one way $\lambda(x_i)$ could not be the best response of the entrant with neighbors at x_i and x_{i+1} is that $i+1$ has not taken his location *and* threatens to locate the closer to the entrant the closer the entrant's location to $\lambda(x_i)$. Assume that the entrant believes this threat and that his best response would be to locate at some $y < \lambda(x_i)$.

To see that this threat is empty in equilibrium (i.e. even if i played his best response to this threat, the threat would in turn not be a best reply), consider the last entrant, say l, to the right of our entrant. Clearly, l's best response will be to locate at $\lambda(x_{l-1})$, where x_{l-1} is the last entrant's left-hand neighbor (which may or may not be $i+1$). Anticipating this, $l-1$ recognizes that l's best responses increases in his own location, and thus he chooses the largest location which allows him to deter entry to his left. By iteration, we see that $i+1$'s best response is to locate at the largest location that deters entry to the left. Thus, the threat is empty on equilibrium and the best response is unique. ■

Intuitively, the optimal location of firm i is driven solely by the location of its left-hand neighbor, y say. As for the right-hand neighbor, either it is already there in which case, because of $f' > 0$, firm i optimally moves to the right as far as possible without inviting further entry to its left, which is at $\lambda(y)$. When i anticipates its right-hand

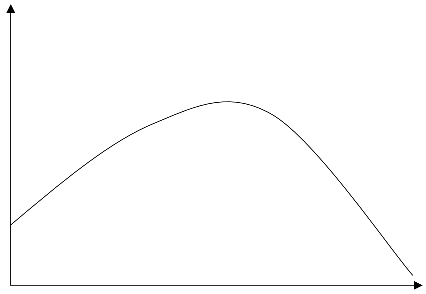

Figure 5: Hump shaped f

neighbor to be a subsequent entrant, then a fortiori $\lambda(y)$ is optimal for firm i, because its future right-hand neighbor will also optimally locate at $\lambda(\lambda(y))$, so that i pushes this firm as far to the right as possible.

We now turn the case of monotone decreasing densities. Let m be the largest integer such that

$$\lambda^0 < \rho^m.$$

Existence and uniqueness of m follow from the monotonicity of $\rho(.)$.

Theorem 5 *Assume $f'(x) < 0$ for all $x \in [0,1]$. Then, the set of locations occupied in any equilibrium is unique. It is*

$$\{\lambda^0, \rho^m, \rho^{m-1}, .., \rho^1\},$$

and the number of firms who enter in equilibrium is $m+1$.

The proof is completely analogous to the one of Theorem 4 and therefore omitted.

Non-monotone densities The previous results enable us to also solve the case of hump-shaped quasiconcave density functions such as depicted in Figure 5. These are functions that initially increase and then decrease. Let r be such that $f'(\lambda^r) \geq 0$ and $f'(\lambda^{r+1}) < 0$. Analogously, let s be such that $f'(\rho^s) \leq 0$ and $f'(\rho^{s+1}) > 0$. Two cases can

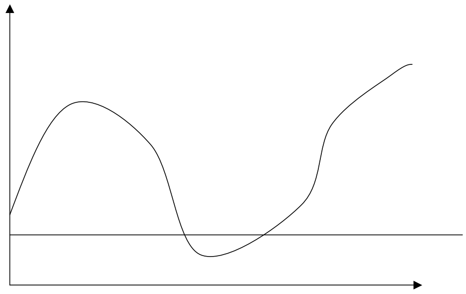

Figure 6: A function that is not quasiconcave

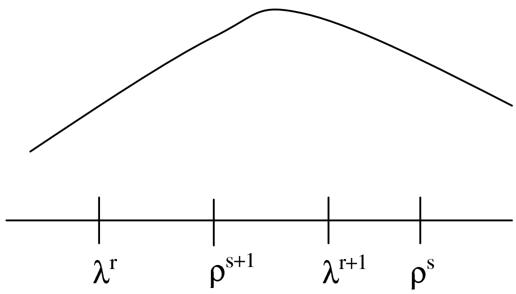

Figure 7: Case I: $\lambda^r < \rho^{s+1} \Leftrightarrow \lambda^{r+1} < \rho^s$.

occur which are depicted in Figures 7 and 8: In Case I, $\lambda^r < \rho^{s+1} \Leftrightarrow \lambda^{r+1} < \rho^s$ holds, while in Case II we have $\rho^{s+1} < \lambda^r \Leftrightarrow \rho^s < \lambda^{r+1}$.

Theorem 6 *Let f be hump-shaped quasiconcave and differentiable everywhere. Then the set of locations occupied in any equilibrium is unique.*

In Case I, it is

$$\{\lambda^0, .., \lambda^r, \hat{x}(\rho^{s+1}, \lambda^{r+1}, \lambda^r, \rho^s), \rho^s, .., \rho^1\},$$

and the number of firms who enter in equilibrium is $r + s + 2$.

In Case II, the set of locations occupied in any equilibrium is

$$\{\lambda^0, .., \lambda^r, \rho^s, .., \rho^1\},$$

and the number of firms who enter in equilibrium is $r + s + 1$.

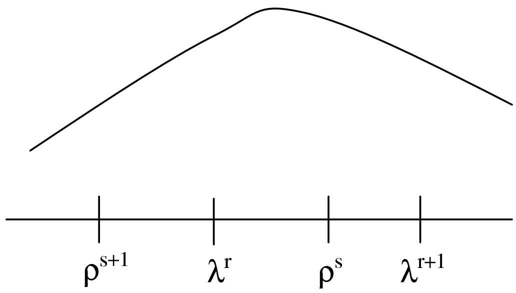

Figure 8: Case II: $\rho^{s+1} < \lambda^r \Leftrightarrow \rho^s < \lambda^{r+1}$.

The difference between the two cases is whether there is an entrant "underneath the hump" or not. This depends solely on whether the highest λ-location on the increasing part of f (i.e. λ^r) and the smallest ρ-location on the decreasing part (ρ^r) are by themselves entry-deterring or not. Global differentiability of f ensures uniqueness of this entrant's location $\hat{x}(\cdot)$.

Proof: The proof is based on Theorems 4 and 5 and is straightforward. On the increasing part, all best responses are independent of the location of the right-hand neighbor. Hence $\{\lambda^0, .., \lambda^r\}$ follows. On the decreasing part, best responses are independent of the left-hand neighbor's location, whence $\{\rho^s, .., \rho^1\}$ follows. Thus, whether there is, or will be, an additional firm between λ^s and ρ^r does not affect the location choice of any of the other firms. An additional firm enters only in case I, in which case it is optimal for this firm to deter entry (see Lemma 6), which it optimally does by locating at $\hat{x}$. ■

5.2 Non-quasiconcave density functions

Consider now density functions that have both peaks and troughs such as the one depicted in Figure 6. For simplicity, we shall assume that peaks and troughs follow each other not too frequently. In particular, we assume that the mass between a peak and a trough is no less than $2T$. This guarantees that between a peak and trough at least one firm enters.

Assume f be U-shaped satisfying $f'(x) < 0$ for all $x < O_1$, $f'(O_1) = 0$ and $f'(x) > 0$ for all $x > O_1$. All other non-quasiconcave functions satisfying our above "normality" assumptions will be easily understood once we understand this case.

Let

$$\rho_x^1 \equiv \rho(x), \quad \rho_x^2 \equiv \rho(\rho_x^1), \quad \text{and} \quad \rho_x^{j+1} \equiv \rho(\rho_x^j)$$

and

$$\lambda_x^1 \equiv \lambda(x), \quad \lambda_x^2 \equiv \lambda(\lambda_x^1), \quad \text{and} \quad \lambda_x^{k+1} \equiv \lambda(\lambda_x^k).$$

Let z be the largest location occupied in an equilibrium satisfying $z \leq O_1$ and let y be the smallest location occupied in this equilibrium with $y \geq O_1$. Notice that this includes the case $z = y$. Moreover, let m and n be such that

$$\rho_z^{m+1} < \lambda^0 < \rho_z^m \quad \text{and} \quad \lambda_y^n < \rho^1 < \lambda_y^{n+1}.$$

Lemma 7 *The locations occupied in this equilibrium are*

$$\{\lambda^0, \rho_z^n, .., \rho_z^1, z, y, \lambda_y^1, .., \lambda_y^n, \rho^1\}.$$

A few remarks are in order. The lemma is illustrated in Figure 9. It is both straightforward to prove and quite powerful. Essentially, it implies that once we have determined z and y we have determined the equilibrium locations. Put differently, the equilibrium locations are determined from the inside, i.e. from O_i. Notice also the following, almost trivial or tautological result:

Corollary 4 *Assume that y is occupied prior to z in equilibrium. Then, y must be such that the firm entering at z deters further entry between them.*

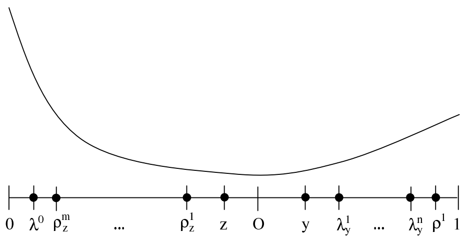

Figure 9: Equilibrium locations (Lemma 7).

Proof of Lemma 7: Given that z and y are occupied in equilibrium, we have a game with decreasing density to the left of z and one with increasing density to the right of y. Thus, Theorems 4 and 5 apply. ■

Assuming (without any real loss of generality) that y is taken prior to z, y must be such that z deters further entry (Corollary 4). Generally speaking, the optimal location for y is then the location that maximizes his profit subject to the constraint that, as z subsequently enters, z deters further entry.

Let x_1, x_2 be two occupied locations with $x_1 < O_1 < x_2$ (where it will be recalled that O_1 is such that $f'(O_1) = 0$) and let q, s be such that

$$x_1 \leq q < s \leq x_2.$$

Lemma 8

$$x^*(x_1, x_2) \in \{x_1^+, x_2^-\}.$$

Proof: The length of the interval captured by the entrant is always $\frac{x_1 - x_2}{2}$ independent of his location. Now let the entrant locate at one end of the interval (say, at x_1^+) and let him contemplate moving marginally towards the middle. Either his profit increases immediately. In this case, however, his profit will keep increasing as he moves further to the right since he keeps losing less on the left and gaining more on the right. Thus, the optimal location will be x_2^- in this situation. Or, the move towards the right will initially involve losses. If this is the case for all positions to the right of x_1^+, then he optimally locates at x_1^+. If eventually the profit starts increasing by moving further right, then it will increase monotonically from there onwards. Hence, the optimal location in this case will be either x_1^+ or x_2^-. ■.

Lemma 9 *No minimum point O_i is ever occupied in equilibrium.*

Proof. Suppose to the contrary that O_i is occupied by, say, firm k. From Lemma 7 we know that its left- and right-hand neighbors are (or will be), respectively, at $\rho(O_i)$ and $\lambda(O_i)$. Without loss of generality, assume $f\left(\frac{\lambda(O_i)+O_i}{2}\right) \geq f\left(\frac{\rho(O_i)+O_i}{2}\right)$. By moving marginally to the right, the profit of k increases: It gains more on the right than it loses on the left. Note that this is true independent of whether the right-hand neighbor is already there or not: If it is not there, the move to the right to some $x > O_i$ k pushes its future right-hand neighbor to $\lambda(k)$. Though the move to the right will attract entry to the left at $\rho(x)$, the loss due to this entry will be smaller than the gain to the left. Thus, O_i cannot be optimal. If $f\left(\frac{\lambda(O_i)+O_i}{2}\right) < f\left(\frac{\rho(O_i)+O_i}{2}\right)$, analogous arguments apply for the opposite direction. ■

Lemma 10 *(i) $\hat{x}(q, s, x_1, x_2) \in \{q, s\}$ and (ii) $x^*(q, x_2) = q^+$ implies $\hat{x}(q, s, x_1, x_2) = q^+$.*

Proof: We first show (ii). Observe first that $x^*(q, x_2) = q^+$ implies $\hat{x}(q, s, q, x_2) = q^+$: The problem that has $x^*(q, x_2)$ as solution is the relaxed problem that yields $\hat{x}(q, s, q, x_2)$ as solution. But since the solution to the relaxed problem is in the choice set for the more constrained problem, the solution of the latter must equal the solution of the former. Second, the problems yielding $\hat{x}(q, s, x_1, x_2)$ and $\hat{x}(q, s, q, x_2)$ differ only with respect to the location of the closest lefthand neighbor. Moving the closest lefthand neighbor away from q to x_1 will only make low (or left) locations more attractive compared to high locations. Hence, $\hat{x}(q, s, x_1, x_2) = \hat{x}(q, s, q, x_2)$ follows.

Part (i) follows by arguments analogous to those of the proof of Lemma 8. ■

Lemma 10 has a noteworthy implication:

Lemma 11 *A sufficient condition for y to induce entry deterrence by the subsequent entrant is $x^*(\rho(y), y) = \rho(y)^+$.*

Proof: Assume that x_1 is not too far away from y, i.e. $x_1 \geq \rho_y^2$, the player choosing z knows that he will invite further entry if he locates at some $z < \rho(y)$. The next (and last) entrant would then choose $\hat{x}(\rho(y), \lambda(z), z, y)$ and thereby deter further entry. From Lemma 10, we know that $\hat{x}(\rho(y), \lambda(z), z, y) = \rho(y)^+$. In this case, though, the firm at z would be caught within two neighbors inside the $[x_1, \lambda(x_1)]$-interval, where it cannot break even (Corollary 1).■

The condition is only sufficient, though: If z decided to accommodate entry, entry would occur on the wrong side of O_1 (for him). Thus, he would never do so. However, even if entry occurred and the other side of O_1, he may optimally choose not to accommodate further entry simply because this does not pay - e.g., assume that f is almost uniform in the neighborhood of O_1. Then z would always prefer to deter entry if possible: Keep his righthand neighbor temporarily fixed at y. By moving left, he would gain essentially no more on the left than what he loses on the right. On top of that, for f almost uniform, his righthand neighbor would move closer almost one-by-one, hence he would strictly do worse. On the other hand, if $\mid f' \mid$ is very large on both sides of O_1 and if $f(O_1)$ is very small, one would expect the sufficient condition also to be necessary. Hence, we first look at the case, where

$x^*(\rho(y), y) = \rho(y)^+$ is both necessary and sufficient to deter further entry.

Observe that this will then yield our condition

$$F\left(\frac{\rho(y)+y}{2}\right) - F(\rho(y)) \geq F(y) - F\left(\frac{\rho(y)+y}{2}\right).$$

Lemma 12 *If $x^*(\rho(y), y) = y^-$ strictly (meaning: $\rho(y)^+$ is not a best reply), there is a $z < \rho(y)$ such that $\hat{x}(\rho(y), \lambda(z), z, y) = \lambda(z)$.*

Proof: Under the assumption $x^*(\rho(y), y) = y^-$, $\lambda(z) \in x^*(z, \lambda(z))$ holds because of continuity for z close to but below $\rho(y)$. Now adding the restriction that the location must be no smaller than $\rho(y)$ will obviously not affect the solution if the unrestricted solution is already above $\rho(y)$. On the other side, moving the righthand neighbor further away from $\lambda(z)$ would make the unrestricted optimal location larger than $\lambda(z)$. Hence, the restricted optimal location $\hat{x}(\rho(y), \lambda(z), z, y)$ will indeed be $\lambda(z)$. ■

Call the density $f(x) = 2 - 4x$ for $x \leq \frac{1}{2}$ and $f(x) = -2 + 4x$ else *two-sided triangular*. For this density, $x^*(\rho(y), y) = \rho(y)^+$ is indeed both necessary and sufficient to deter further entry for $y \approx \tilde{y}$: Suppose $x^*(\rho(y), y) = y^-$. If the best reply of the entrant after z is to locate at $\lambda(z)$ for z close to, but below $\rho(y)$, then the firm at z loses almost nothing in the middle (since $f(\frac{z+\lambda(z)}{2}) \approx 0$) while he has strictly positive gain on the left of more than $\frac{f(z)}{2} > 0$.

Theorem 7 *For the two-sided triangular density and for $T < \frac{1}{4}$,*

$$\lambda^0 \equiv \frac{1-\sqrt{1-2T}}{2} \quad \textit{and} \quad \rho^1 \equiv \frac{1-\sqrt{1-2T}}{2}$$
$$y^* \equiv \frac{1+\sqrt{2T}}{2} \quad \textit{and} \quad z^* \equiv \frac{1-\sqrt{2T}}{2}.$$

The locations occupied in equilibrium are

$$\{\lambda^0, \rho^m_{z^*}, .., \rho^1_{z^*}, z^*, y^*, \lambda^1_{y^*}, .., \lambda^n_{y^*}, \rho^1\}.$$

Proof:

The distribution function is $F(x) = 2x - 2x^2$ for $x < \frac{1}{2}$. A firm entering at y will invite entry on $\rho(y)$ instead of close to y only if $\rho(y)$ is the weakly more attractive location, which is the case if and only $\frac{\rho(y)+y}{2} \leq \frac{1}{2} \Leftrightarrow \rho(y) \leq 1 - y$. Since this is a necessary and sufficient condition for further entry deterrence by the subsequent entrant, this condition will be met. Hence, $\rho(y)$ will be such that

$$F\left(\frac{\rho(y)+y}{2}\right) - F(\rho(y)) = \rho(y) + y - 2\left(\frac{\rho(y)+y}{2}\right)^2 - 2\rho(y) + 2\rho(y)^2 = T.$$

Solving for $\rho(y)$ yields as relevant solution

$$\rho(y) = \frac{1 + y - \sqrt{4y^2 - 4y + 1 + 6T}}{3}.$$

There being no use for the firm choosing y to locate such that $\rho(y) < 1-y$, the equilibrium y is such that $\rho(y) = 1 - y$. Solving this quadratic equation yields

$$y^* = \frac{1 + \sqrt{2T}}{2}$$

as relevant solution, whence

$$z^* = \rho(y^*) = \frac{1 - \sqrt{2T}}{2}$$

follows. To determine λ^0, observe that λ^0 satisfies $F(\lambda^0) = 2\lambda^0 - 2(\lambda^0)^2 = T$, yielding the value in the theorem as relevant solution. By symmetry, $\rho^1 = 1 - \lambda^0$ follows. Last, $\lambda^0 < \rho(y^*)$ has to hold. It is easily seen that this is true if and only if $T < \frac{1}{4}$. ■

For $\frac{1}{4} \leq T < \frac{1}{2}$, two firms will enter in equilibrium, which is similar to PV. However, for T just below $\frac{1}{4}$ four firms enter, whereas in PV only three do so. [20]

Let O_i be the i-th minimum of the density f, ordered from left to right, with $i = 1, 2, ...$ That is, each O_i satisfies

$$f'(O_i) = 0 \quad \text{and} \quad f''(O_i) > 0.$$

[20] "All the while you're waiting for a bus, and then all of a sudden there come two."

Let $M \geq 1$ be the number of minima of f and define the i-th partgame of our game as follows:

Definition 3 (Partgame) *The first partgame is the game in the interval*

$$[0, O_1],$$

where 0 and 0_1 *are unoccupied borders. Similarly, the* $M+1$*-th partgame is the game in the interval*

$$[O_M, 1],$$

and in general, the i*-th partgame is the game in the interval*

$$[O_{i-1}, O_i],$$

where O_{i-1}, O_i, O_M *and 1 are unoccupied borders.*

The following result is immediate:

Lemma 13 *In each partgame, the function* f *is quasiconcave.*

Therefore, each partgame has a unique set of locations occupied in equilibrium.

Definition 4 (Trough-symmetry) *The function* f *is called trough-symmetric if it is symmetric in a* $\rho-\lambda$*-neighborhood around every minimum point* O_i*. That is,* f *is trough-symmetric if and only if*

$$f(O_i - x) = f(O_i + x)$$

for any

$$x \in [\rho^B(O_i), \lambda^B(O_i)]$$

and every for every $O_i, i = 1, .., M,$ *where* ρ^B, λ^B *denotes border points.*

Theorem 8 (Separation) *Let f be trough-symmetric. Then the full game has a Nash equilibrium in which the same locations are occupied as the union of the sets of locations occupied in the Nash equilibrium of every partgame.*

A few comments are in order. First, the theorem says nothing about uniqueness or subgame perfection. These are issues we are currently working on. Second, trough-symmetry is admittedly a very specific assumption, but other than that the theorem is remarkably general and its proof remarkably simple. Third, trough-symmetry is only a sufficient condition: The theorem would also hold under the slightly weaker assumption $O_i - \rho^B(O_i) = \lambda^B(O_i) - O_i$.

Proof: We are going to argue that for trough-symmetric functions, $\rho^B(O_i)$ and $\lambda(O_i)$ are mutually best responses. Once this is shown, the Theorem follows immediately from the previous results. So, suppose some firm is located at $\lambda^B(O_i)$. Observe then that any location $y \in [\rho^B(O_i), \lambda^B(O_i))$ will deter entry in between: For $y > \rho^B(O_i)$, $x^*(y, \lambda^B(O_i)) = \lambda^B(O_i)^-$. But $\pi^* < T$ since $\frac{y+\lambda^B(O_i)}{2} > O_i$ because of trough-symmetry. Since moving away from the middle without attracting entry is always beneficial, $\rho^B(O_i)$ dominates any interior location. Notice then that at $y = \rho^B(O_i)$ the firm nets exactly T to the right, again because of trough-symmetry. Thus, this is a best response. Mutuality of best responses follows from symmetry. ■

6 Settlement

In general, for $F(\cdot)$ and a sequence of entry given, the order of settlement is pinned down, the earlier entrants grasping the larger profits, except when two or more equilibrium profits are the same. Without additional assumptions on $F(\cdot)$, however, it will in general not be possible to determine the ordering of equilibrium profits. Nonetheless, some results can be derived under fairly general conditions. In the following, we denote by $\pi(y)$ the equilibrium profit of a firm at (equilibrium) location y.

6.1 Monotone densities

Recall from Theorem 4 that when f is increasing on $[0,1]$, the two rightmost equilibrium locations are λ^n and ρ^1, respectively. Analogously from Theorem 5, when it is decreasing, the two leftmost equilibrium locations are λ^0 and ρ^m, respectively:

Lemma 14 *(i) If $f'(x) > 0$ for all $x \in [0,1]$, then*

$$\pi(\rho^1) > \pi(\lambda^n),$$

such that ρ^1 will be occupied prior to λ^n in any equilibrium.

(ii) If $f'(x) < 0$ for all $x \in [0,1]$, then

$$\pi(\lambda^0) > \pi(\rho^m),$$

such that λ^0 will be occupied prior to ρ^m in any equilibrium.

Proof.

As for part (i), note first that the firm at ρ^1 earns T to its right by definition of ρ^1. As for the firm at λ^n, note that when $f' > 0$, $x^*(y, \lambda(y)) = \lambda(y)^-$ so that $\pi(\lambda(y)^-, y, \lambda(y)) = T$. Thus, the firm at λ^n earns T to its left so that differences in their profits can only accrue from differences in earnings between λ^n and ρ^1. In terms of distances, both grasp exactly $\frac{\rho^1+\lambda^n}{2}$. However, the density over the share grasped by the firm at ρ^1 being larger than for the share catered by the firm at λ^n, it follows that the firm at ρ^1 earns strictly more. The proof for part (ii) is completely analogous. ■

We now compare $\pi(\rho^1)$ and $\pi(\lambda^0)$ (recall from Theorem 3 that the locations λ^0 and ρ^1 will be occupied in *any* equilibrium).

To this end, let $\overline{T}(j+2)$ be the largest fixed cost such that exactly $j+2$ firms enter where $j = m, n$. Thus, when f is increasing, we have $\lambda^n < \rho^1$ for all $T \leq \overline{T}(n+2)$. Analogously, when f is decreasing, $\rho^m > \lambda^0$ holds for all $T \leq \overline{T}(m+2)$.

Lemma 15 *(i) If $f' > 0$ and T sufficiently close to $\overline{T}(n+2)$, then*

$$\pi(\lambda^0) > \pi(\rho^1),$$

such that location λ^0 will be occupied prior to location ρ^1.

(ii) If $f' < 0$ and T sufficiently close to $\overline{T}(m+2)$,

$$\pi(\lambda^0) < \pi(\rho^1),$$

such that location λ^0 will be occupied after location ρ^1.

Proof. sketch: As for part (i), for $T = \overline{T}(n+2)$, $\lambda^n = \rho^1 - \varepsilon$. Hence, as T approaches $\overline{T}(n+2)$, $\pi(\rho^1)$ approaches T. On the other hand, the firm at λ^0 will earn T to its left and something strictly larger than zero to its right if T approaches $\overline{T}(n+2)$. ■

6.2 Concave densities

Under the additional restriction that f is concave, we can also say something about the ordering of the profits of the firms locating in $[0, \lambda^{n-1}]$ when f is increasing and of those locating in $[\rho^{m-1}, \rho^0]$ when f is decreasing.

Lemma 16 *(i) If $f'(x) > 0$ and $f''(x) \leq 0$ for all $x \in [0,1]$, then*

$$\pi(\lambda^{n-1}) > ... > \pi(\lambda^0)$$

holds, i.e the settlement of $\{\lambda^0, ..., \lambda^{n-1}\}$ occurs from the right to the left.

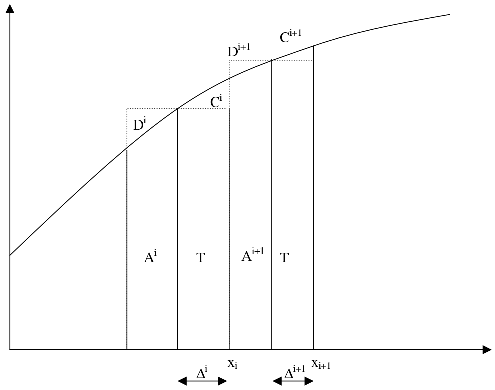

Figure 10: $\pi^i = T + A^{i+1}$.

(ii) If $f'(x) < 0$ and $f''(x) \leq 0$ holds for all $x \in [0,1]$, then

$$\pi(\rho^{m-1}) > ... > \pi(\rho^1)$$

holds, i.e. the settlement of $\{\rho^{m-1}, ..., \rho^1\}$ occurs from the left to the right.

Proof: We only prove the lemma for the $f' > 0$, the case with $f' < 0$ being completely analogous. Consider Figure 10 to see that the equilibrium profit of the firm at location x_i is equal to the sum of two areas: To the left, it gets an area of size T, and to the right an area of size A^{i+1}, which is smaller than T. So

$$\pi(x_i) = T + A^{i+1}.$$

Let $\Delta^i := \Delta(x_{i-1}, x_1)$, i.e. half of the distance between the equilibrium locations x_{i-1} and x_i. Because $f(x)$ increases in x, $\Delta^{i+1} < \Delta^i$ holds. We are now going to show that for the areas A^i the following holds: $A^i < A^{i+1}$ for any $i \geq 1$. Since $\pi(x_i) = T + A^{i+1}$, this will then complete the proof of the Lemma.

Observe first that $A^i = T - C^i - D^i$. So $A^i < A^{i+1}$ will hold if we can show that $D^{i+1} < D^i$ and $C^{i+1} < C^i$ holds. Define $\tilde{f}_i(x) \equiv f(x) - f(x_i - \Delta^i)$ for $x \in [x_i - \Delta^i, x_i]$. Clearly, $\tilde{f}_i' > 0$ and $\tilde{f}_i'' \leq 0$ holds. For any $y < \Delta^{i+1}$ this implies

$$\tilde{f}_i(x_i - \Delta^i + y) \geq \tilde{f}_{i+1}(x_{i+1} - \Delta^{i+1} + y). \tag{3}$$

Observe next that

$$C^i = \int_{x_i-\Delta^i}^{x_i} \tilde{f}_i dx > \int_{x_i-\Delta^i}^{x_i-\Delta^i+\Delta^{i+1}} \tilde{f}_i dx \geq \int_{x_{i+1}-\Delta^{i+1}}^{x_{i+1}} \tilde{f}_{i+1} dx = C^{i+1}.$$

The first and last equality are identities. The first inequality is due to the fact that $\Delta^i > \Delta^{i+1}$, and the weak inequality follows from (3). Thus, $C^{i+1} < C^i$ is established.

Similarly, define $\hat{f}_i(x) \equiv f(x_i) - f(x)$ for $x \in [x_i - 2\Delta^i, x_i - 2\Delta^i]$. For any $y < \Delta^{i+1}$,

$$\hat{f}_i(x_i - 2\Delta^i + y) \geq \hat{f}_{i+1}(x_{i+1} - 2\Delta^{i+1} + y) \tag{4}$$

holds. Observe then that

$$D^i = \int_{x_i-2\Delta^i}^{x_i-\Delta^i} \hat{f}_i dx > \int_{x_i-2\Delta^i}^{x_i-2\Delta^i+\Delta^{i+1}} \hat{f}_i dx \geq \int_{x_{i+1}-2\Delta^{i+1}}^{x_{i+1}-\Delta^{i+1}} \hat{f}_{i+1} dx = D^{i+1}.$$

The proof for part (ii) is completely analogous.. ■

The following remarks are in order. First, the condition $f'' \leq 0$ is only sufficient since the result would hold if f were somewhat convex, but not too much so. Second, and more importantly, observe that the lemma is valid for linear densities. In particular, it holds for any density with slope $f'(x) = \varepsilon$, where $\mid \varepsilon \mid \neq 0$ is arbitrarily small. Consequently, for any such density the settlement will be generically unique, i.e. for almost every T there is a unique sequence of settlement. This contrasts sharply with the uniform case, where (but for at most three locations) the sequence of settlement is arbitrary in any equilibrium. This corroborates our claim in the Introduction that the uniform case is a very special one.

Combining Lemmas 15 and 16, the following result is immediate:

Corollary 5 *(i) Let $f' > 0, f'' \leq 0$ and let T be close to $\overline{T}(n+2)$. Then,*

$$\pi(\lambda^{n-1}) > .. > \pi(\lambda^1) > \pi(\lambda^0) > \pi(\rho^1) > \pi(\lambda^n),$$

i.e., the sequence of settlement will be $\lambda^{n-1}, ..., \lambda^1, \lambda^0, \rho^1, \lambda^n$.

(ii) Let $f' < 0, f'' \leq 0$ *and let* T *be close to* $\overline{T}(n+2)$*. Then,*

$$\pi(\rho^{m-1}) > .. > \pi(\rho^2) > \pi(\rho^1) > \pi(\lambda^0) > \pi(\rho^m),$$

i.e., the sequence of settlement will be $\rho^{m-1}, ..., \rho^2, \rho^1, \lambda^0, \rho^m$.

Notice the following "non-monotonicity" property of equilibrium profits: As for part (i), the neighboring locations λ^{n-1} and λ^n are the most and least lucrative ones, respectively. The reason is that the location λ^n is the only one whose right-hand neighbor (ρ^1) is not determined by a λ-distance, but from the right boundary of the product spectrum. As T increases, the distance between λ^n and ρ^1 declines, so that this firm's payoff to the right vanishes, while it is strictly positive payoffs for all other equilibrium locations. An analogous argument applies for part (ii).

Given Lemma 16, a (albeit incomplete) characterization of the sequence of settlement for concave and hum-shaped densities is also at hand (recall that the equilibrium locations for this case were determined in Theorem 6).

Corollary 6 *Let* f *be concave and hump-shaped. Then, the sequence of settlement of the locations*

$$\{\lambda^0, .., \lambda^{r-1}\}$$

occurs from the right to the left and the sequence of settlement of the locations

$$\{\rho^{s-1}, .., \rho^1\}$$

occurs from the left to the right.

The result follows directly from Lemma 16 and requires no separate proof. The characterization is incomplete as it does not say anything about (i) the profits at locations

λ^r and ρ^m (and eventually $\hat{x}$) and (ii) it does not rank the locations to the left relative to those on the right. Nonetheless, the result is quite remarkable because it says that under fairly general conditions, the sequence of settlement is, essentially, "inside-out", and thus in strong contrast to the "outside-in"- principle as claimed by Prescott and Visscher (1977, fn. 5).

It is also worth mentioning that an analogue to Lemma 15 for a general hump-shaped density cannot be obtained because a T such that λ^r and ρ^s are just adjacent to each other need not exist: One side of the hump shaped density may simply be more attractive than the other. Consequently, as, say, λ^r is so large that $f'(\lambda^r) = 0$, $f'(\rho^s) < 0$ may still hold. However, the following result is readily established:

Lemma 17 *Assume that f is hump-shaped, concave and symmetric and let T be close to $\overline{T}(r + s + 1)$. Then:*

$$\pi(\lambda^{r-1}) = \pi(\rho^{s-1}) > ... > \pi(\lambda^0) = \pi(\rho^1) > \pi(\lambda^r) = \pi(\rho^s),$$

and the settlement starts either at λ^{r-1} or ρ^{s-1} and then goes outwards. The locations λ^r and ρ^s are occupied last.

The proof follows the one of Lemma 15 and is therefore omitted.

7 Discussion

7.1 Comparison with the uniform case

We now consider the special case where consumers are uniformly distributed (as analyzed in Prescott and Visscher (1977)), and compare it to our general results.

When $F(y) = y$ and absent subsequent entry, a firm's payoff in some interval (L, R)

is constant, and simply equal to the range of its customer base:

$$\pi(x, L, R) = \frac{x+R}{2} - \frac{x+L}{2} = \frac{R-L}{2} = \Delta(L, R)$$

Therefore, $X^*(L, R) = (L, R)$ from which it immediately follows that **any** entry-deterring location in the interval (L, R), if it exists, will automatically be an optimal one. As seen above, this is not generally true, and thus the uniform case is special in this important aspect.

We use the subscript U to denote variables and functions for the uniform. Note first that $\lambda_U^0 = T$ and $\rho_U^1 = 1 - T$. Furthermore, one easily calculates that

$$\lambda_U(y) = y + 2t \quad \text{and} \quad \rho_U(y) = y - 2T.$$

For a given interval (L, R), it follows that

$$\lambda_U(L) \geq R \Leftrightarrow R - L \leq 2T,$$

so that profitable entry is not possible. Furthermore,

$$\lambda_U(L) > \rho_U(R) \Leftrightarrow R - L \in (2T, 4T],$$

in which case the first entrant will optimally enter anywhere in $(R-2T, L+2T)$, thereby forestalling further entry. Note that this firm is indifferent between any entry-deterring location, but strictly prefers any of these to any other location (which would yield the same payoff *absent* further entry, but would invite further entry, so that the first firm is strictly worse off). Finally,

$$\lambda_U(L) < \rho_U(R) \Leftrightarrow R - L > 4T,$$

in which case, the first entrant cannot forestall further entry, and by locating either at $L + 2T$ or $R - 2T$ optimally deters subsequent entry at least with respect to one side.

Before presenting the equilibrium configuration, the following notation is useful: Define $n := \frac{1}{2T}$, and denote by n^I the integer part of n. Furthermore, denote by $\#_U$ the number of firms entering in equilibrium when F is uniform.

Theorem 9 (Uniform distribution) *Suppose F is uniform:*

(i) If $T \in [\frac{1}{4}, \frac{1}{2})$, then $\#_U = 2$ and their locations are

$$\{\lambda^0 = T, 1 - T = \rho^1\}.$$

Both active firms attain the same profit (equal to $\frac{1}{2}$).

(ii) If $T < \frac{1}{4}$ and $2n$ is an even integer, then $\#_U = n$, and their locations are

$$\{\lambda^0 = T, 3T, ..., 1 - 3T, 1 - T = \rho^1\}.$$

All active firms attain the same profit (equal to $2T$).

(iii) If $T < \frac{1}{4}$ and n is not an even integer, then $\#_U = n^I + 1$, and their locations are

$$\{\lambda^0 = T, 3T...1 - 3T, 1 - T = \rho^1\}.$$

All active firms attain the same profit (equal to $2T$), except the three firms "involved" in the (only) small interval the size of which is smaller than $4T$; their profit is strictly less than $2T$.

Note first that a continuum of possible equilibrium configurations arises under part (iii), since the location of the last entrant in the "small" interval is indeterminate.[21]

Moreover, as in the general model, the equilibrium configuration is unaffected by the sequencing in the sense that it does not matter which equilibrium location is occupied

[21] Prescott and Visscher (1977) assume that it locates at the midpoint $\frac{R+L}{2}$.

when. This contrasts sharply with the case of non-uniform distributions, where the order of settlement is generally pinned down as noted above.

Last, but not least, the number of firms entering the market in the uniform case is also weakly lower than under any other distribution, as we will now show.

Theorem 10 *For any distribution* $F(\cdot)$, $\# \geq \#_U$.

Proof. It will be convenient to re-label the uniform distribution function to $\mu(y) = y$ to better distinguish it from general distribution functions $F(\cdot)$.

Recall first from part (iii) of Lemma 3 that $F(\lambda(L)) - F(L) \leq 2T$ and $F(R) - F(\rho(R)) \leq 2T$. Since $\mu(\lambda_U(L)) - F(L) = 2T = \mu(R) - \mu(\rho(R))$, we have $\lambda(L) \leq \lambda_U(L)$ and $\rho_U(R) \leq \rho(R)$.

Moreover, we already know that there cannot be more entry in the uniform when the market size of given interval (L, R) is not too large: Either $\rho_U(R) \leq L < R \leq \lambda_U(L)$, in which case no more entry in the uniform case, or $L < \lambda_U(L) < \rho_U(R) < R$ in which case one firm enters under in the uniform case, while either one or two enter in the general case (see part (ii) of Theorem 1).

Thus, the only possibility for more entrants in the uniform case is in "large" intervals. From Theorem 3, we know that λ^0 and ρ^1 will be occupied under any distribution. Although in general, $\lambda_U^0 \neq \lambda^0$ and $\rho_U^1 \neq \rho^1$ note that, by definition, $\mu(\lambda_U^0) = F(\lambda^0) = T$ and $\mu(\rho_U^1) = F(\rho^1) = 1 - T$.

Since $\mu(\lambda_U^{j+1}) - \mu(\lambda_U^j) = 2T$, but $F(\lambda^{j+1}) - F(\lambda^j) \leq 2T$ for any $j > 0$, it follows that in the uniform case, it takes (weakly) "fewer steps" until an endgame is reached (if it is reached at all). ■

7.2 Robustness, Entry Deterrence, and Profitability

7.3 Preference externalities

Assume there are two groups of consumers, say, whites and blacks. Assume also that whites have on average preferences for, say, newspapers that are on the left end of the [0,1]-interval while blacks are mostly located close to the right end. Moreover, let T be such that exactly two firms enter. Clearly, λ^0 will be the location of the overwhelmingly white newspaper and ρ^1 the location of the black newspaper.

Now add a positive mass of white consumers to the market whose preferences are distributed identically to those of the already present whites. If the added mass of consumers is not too large, the equilibrium number of firms will still be two. If no whites have bliss points in $[\rho^1, 1]$, then the equilibrium location of the black newspaper will not change. The equilibrium location of the white newspaper, however, will move to the left since there are now more consumers in the interval $[0, \lambda^0]$ and entry deterrence requires the white newspaper to keep the mass to its left at T. If some blacks previously read the white newspaper, these will be worse off. Either they now have to travel a longer distance to the white paper or they switch from the white to the black paper, in which case they travel a longer distance to the right. Hence, there can indeed be preferences externalities.

8 Conclusions

In this paper, we have studied dynamic location games, in which firms enter sequentially, pay a fixed cost upon entry and cannot change locations once they are chosen. Under fairly general assumptions, we have shown how to derive the equilibrium locations and the sequence of settlement and that the equilibrium locations and sequence of settlement are generically unique.

Two avenues for further research seem particularly promising. First, we have assumed throughout that, though entry occurs sequentially, there are no costs to later entry other than those that the best locations are occupied first. Thus, in a sense the model we analyzed is a one period model without explicit delay costs. An interesting modification would be to assume that in every period exactly one firm can enter and that all firms discount future payoffs by the discount factor $\delta \leq 1$. This would add an additional feature that seems relevant in reality, namely a trade-off between short-term and long-term profits. The locations most attractive from a short-term perspective are those where the closest neighbors are far away. Unfortunately, these are the locations that are prone to attract additional entrants in the future. Thus, the most attractive locations from a short-term view are least, or less, attractive in the long view.[22]

Second, we have abstracted away from any sort of price competition. Though there are good reasons to do so with respect to consumer prices when these prices are zero to begin with (e.g., broadcasting), these consumers are valid to firms only insofar as they generate advertisement revenue, which we have taken as given. A natural extension would thus be to add a second stage where firms compete in prices for advertisement. The competing firms would then be vertically differentiated by market shares since advertisers will typically not care for the location of the firm but merely for the number of consumers it can reach via this firm. Thus, this extension would combine ideas developed in this paper with models such as Gabszewicz, Laussel, and Sonnac (2001).

[22]The model we have analyzed is a special case of this model with an infinity number of periods and $\delta = 1$: Under these assumptions, no short-term gain will ever outweigh any long-term loss.

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