Tractable Hedging
with Additional Hedge Instruments

This version: March 1, 2007

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Abstract

In an uncertain volatility model where only the stock and the money market account are traded, the upper price bound of a European claim is given by the solution of a Black-Scholes-Barenblatt equation. If an additional hedge instrument is available, the price bound can be tightened. This is also true if the set of admissible strategies is restricted to tractable strategies defined as sums of Black-Scholes strategies. By combining the approach for including additional instruments in the hedge and for finding the optimal tractable hedge, we are able to analyze the structure of the optimal superhedge. We compare both the general strategies and the tractable strategies when an additional call option is traded. The restriction to tractable strategies allows for economic arguments in finding the optimal position in the additional hedge instrument.

Keywords: Stochastic Volatility, Robust Hedging, Tractable Hedging, Uncertain Volatility Model, Additional Hedge Instrument

JEL: G12, G13

1. Introduction

In a complete market, any contingent claim can be replicated by a self-financing portfolio strategy consisting of basis assets, and can thus also be hedged perfectly. In practice, however, markets are not complete due to a lot of reasons. For example, one might think of models with stochastic volatility and stochastic jumps[1], where in the latter case infinitely many traded would be needed to complete the market. Another problem are trading restrictions and the necessary discretization of trading strategies.[2] Recently, great attention is also drawn to the topic of model risk, that is to the problem that the true data–generating process is not known.[3] Model risk poses severe problems when it comes to hedging, since in most cases the investor has to know the true model to determine either a perfect hedge within a complete market model or a hedge which satisfies some optimality criteria (like minimal variance of the hedging error, e.g.) in an incomplete market.

A setup that explicitly takes model risk into account is the uncertain volatility model of Avellaneda et al. (1995). The stock price is assumed to follow a diffusion process with an unknown (but bounded) volatility. The class of possible stochastic processes includes among others the case of a constant volatility as in Black–Scholes, a time–dependent volatility, or stochastic volatility models. For a given claim which is not affine in the price of the underlying, there does not exist a hedging strategy which is perfect with respect to all these processes simultaneously. Instead, we consider superhedging strategies, i.e. self-financing strategies for which the terminal payoff is in all models at least as large as the payoff of the claim. The initial capital needed for such a superhedging strategy is an upper price bound for the claim, and the initial capital needed for the cheapest superhedge is the lowest upper price bound. It can be shown that the cheapest superhedging strategy is equal to a replicating strategy in one of the possible models, the so–called worst–case model. The solution of this problem is firstly given in Avellaneda et al. (1995) and Lyons (1995), and we call the corresponding strategies ALP–hedges in the following. More recently, the problem of superhedging in an uncertain volatility model is also considered in Branger and Mahayni (2006) who introduce an additional constraint to the optimization problem. They restrict the set of admissible strategies to the sum of Black/Scholes–type

[1]Models with stochastic volatility are discussed by Hull and White (1987) and Heston (1993), while Merton (1976) considers a jump-diffusion models. Bakshi et al. (1997), Duffie et al. (2000), Eraker et al. (2003) and Broadie et al. (2005), among others, analyze models with stochastic volatility and jumps.

[2]Discretely adjusted option hedges are e.g. analyzed in Boyle and Emanuel (1980) and Bertsimas et al. (1998).

[3]The impact of model risk on hedging is e.g. studied in Bossy et al. (2000), Bossy et al. (2001), Jiang and Oomen (2001), and Hull and Suo (2002).

strategies, so–called tractable strategies, which are a simple and parsimonious choice if the true model is not known.

The concept of superhedging is often criticized as too expensive. Intuitively, it is clear that a hedge which needs to be effective for a whole set of models simultaneously may afford a very high initial investment. This problem can be mitigated by the introduction of an additional hedge instrument. The expensive superhedge then has to be implemented for the remaining payoff only. Avellaneda and Parás (1996) show how to find the lowest upper price bound in this case. Along the lines of Branger and Mahayni (2006), we introduce the additional constraint that the hedging strategy for the remaining payoff is restricted to a sum of Black/Scholes hedges.

It is clear that the use of an additional instrument in the hedge portfolio will reduce the initial capital. In an uncertain volatility model, the basic intuition can best be explained by considering an additional claim that is convex. Note that the upper price bound of a claim without considering the market price of the additional claim is given by the worst case model which is implied by the claim to be hedged. In particular, convex payoffs are hedged (respectively priced) at the upper volatility and concave payoffs at the lower volatility bound. For payoffs which are piecewise convex and concave, the volatility in the worst case model switches between the upper and the lower volatility depending on the sign of the worst case gamma. By trading in the additional convex claim, the investor buys (sells) its convexity at the market price instead of at the upper (lower) volatility bound, which allows him to reduce the initial capital needed. A similar argument holds for an additional claim which is neither convex nor concave, where the investor trades its curvature at the market price instead of the price implied within the worst–case model.

In this paper, we study the optimal tractable robust hedge if an additional instrument is available. First, we give conditions on the optimal position in the additional instrument and on the remaining payoff. We assume that the additional claim is convex, where we think for example of (highly liquid) ATM–options. Second, we analyze the structure of the optimal hedge. In particular, we are interested in whether the investor uses a long or a short position in the additional instrument, that is whether he cares more about the convexity risk or the concavity risk of the claim to be hedged. We are also interested in the reduction in initial capital he can achieve. Third, we compare our results to the case of ALP–strategies. It turns out that the sign of the optimal position in the additional instrument may differ between tractable hedges and ALP–hedges. The reduction in initial capital may also be larger or smaller for tractable hedges than for ALP–hedges.

We illustrate our findings by two examples. First, we consider hedging a call option. As expected, the position in the additional ATM–option is the larger the more similar the two claims are in terms of their strikes and the larger the upper volatility bound, that is the price of hedging convexity risk. Furthermore, the reduction in the initial capital due to the use of the additional instrument is larger for the ALP–hedge than for the tractable hedge, and the investor takes a smaller position in the additional instrument in case of tractable hedging than in case of the ALP–hedge. The intuition for both these findings is that the original payoff is convex, but that the remaining payoff which has to be hedged after buying the call option is neither convex nor concave. And while the tractable hedge and the ALP–hedge coincide for convex claims, the tractable hedge is more expensive than the ALP–hedge for mixed payoffs. This makes the investor more reluctant when trading the additional claim. The second claim we consider is a bullish vertical spread, which is a mixed payoff from the very beginning. A positive position in the hedge instrument implies that the investor buys convexity at the market price and thus cares more about the convexity of the claim than about the concavity. Interestingly, the optimal position in the additional call can be a short position or a long position, and it may also be optimal to ignore the additional call.

The paper is organized as follows. Section 2 recalls some well known results about the price bounds in an uncertain volatility model with and without the restriction of strategies. Section 3 gives the maximization problem if an additional hedge instrument is available, and analyzes the structure of the optimal tractable robust hedge. Section 4 discusses two examples, a call option and a bullish vertical spread. Section 5 concludes.

2. Price bounds in an uncertain volatility model

Along the lines of Avellaneda et al. (1995) we consider an *uncertain volatility model*. There are two traded assets, a risky asset X and a zero bond B with maturity in T. All prices are already expressed in units of this zero bond, which implies $B \equiv 1$. The asset price X follows a diffusion process with a diffusion coefficient σ which is bounded above by $\sigma_{\max}$ and bounded below by $\sigma_{\min}$. In particular, we have

$$dX_t = X_t \left(\mu_t \, dt + \sigma_t \, dW_t\right) \tag{1}$$

where μ_t and σ_t are non–anticipative functions such that $\sigma_{\min} \leq \sigma_t \leq \sigma_{\max}$, and W is a Brownian Motion under the physical probability measure P.

Since the volatility is unknown and may well be stochastic, the investor cannot implement a perfect hedge for non-affine claims, and the market is incomplete. Instead, we will consider super– (and sub–)hedges in the following, i.e. self–financing strategies which dominate (are domatinated by) the claim to be hedged with probability one in all models with bounded volatility given by Equation (1). In the following, these superhedging strategies are called *robust hedging strategies*.

We deal with European path–independent claims with maturity $T > 0$ and characterize a claim by its payoff–function $h : \mathbb{R} \to \mathbb{R}_+$. The upper price bound for a claim h is the lowest initial capital needed for a self–financing strategy which dominates h, i.e. the initial capital for the cheapest robust strategy. Note that a robust strategy for h dominates the payoff h and thus actually provides a hedge for a short–position in the claim with payoff–function h.

In the unconstrained case, there is no restriction on the set of admissible strategies.[4] The lowest upper price bound u^{ALP} of a payoff–function h is given by[5]

$$u^{\text{ALP}}(t, x; h) := \text{Sup}_{Q \in Q^*} E_t^Q[h(X_T)] \tag{2}$$

where Q^* denotes the class of all probability measures on the set of paths $\{S_t, 0 \le t \le T\}$, such that for some bounded σ_t it holds that

$$dX_t = X_t \sigma_t \, dW_t^*.$$

In Avellaneda et al. (1995) and Lyons (1995) it is shown that this lowest upper price bound is the solution of a Black–Scholes–Barenblatt (BSB) equation.[6] In particular, for all $x \ge 0$ it holds

$$\begin{aligned}
\frac{\partial u^{\text{ALP}}}{\partial t}(t, x; h) + \frac{1}{2}\sigma^2(t, x; h) x^2 \frac{\partial^2 u^{\text{ALP}}}{\partial x^2}(t, x; h) &= 0 \\
u^{\text{ALP}}(T, x; h) &= h(x) \\
\text{where } \sigma(t, x; h) &= \sigma_{\max} 1_{\left\{\frac{\partial^2 u^{\text{ALP}}}{\partial x^2}(t,x;h) \ge 0\right\}} + \sigma_{\min} 1_{\left\{\frac{\partial^2 u^{\text{ALP}}}{\partial x^2}(t,x;h) < 0\right\}}.
\end{aligned}$$

The corresponding cheapest robust hedge is given by a portfolio consisting of ϕ_t^X assets X and ϕ_t^B bonds where

$$\phi_t^X = \frac{\partial u^{\text{ALP}}}{\partial x}(t, X_t; h), \quad \phi_t^B = u^{\text{ALP}}(t, X_t; h) - \frac{\partial u^{\text{ALP}}}{\partial x}(t, X_t; h) X_t, \tag{3}$$

c.f. Avellaneda et al. (1995). In general, the price bound $u^{\text{ALP}}(t, x; h)$ and the corresponding volatility $\sigma(t, x; h)$ have to be determined simultaneously. The solution is given by a Hamilton–Jacobi–Bellman equation and is linked to a stochastic control problem. If h is convex (concave), the problem simplifies considerably. In this case, $\sigma(t, x; h) = \sigma_{\max}$ ($\sigma(t, x; h) = \sigma_{\min}$), and the BSB partial differential equation reduces to a Black–Scholes (BS) equation. Convex (concave) claims are thus hedged and priced at the upper (lower) volatility bound.

[4]The set of admissible strategies is given by all predictable processes $\phi = (\phi^X, \phi^B)$.

[5]See e.g. El Karoui and Quenez (1995).

[6]The existence, uniqueness and smoothness for this equation is analyzed in detail in Vargiolu (2001).

We now restrict the set of admissible strategies to tractable strategies along the lines of Branger and Mahayni (2006). A *tractable robust hedge* for h is represented by $(\overline{h}, \underline{h})$ where $\overline{h}$ is convex and $\underline{h}$ is concave and where

$$\overline{h}(y) + \underline{h}(y) \quad \geq \quad h(y) \quad \forall y \geq 0.$$

The tractable robust hedge itself is given by the sum of a BS-hedge for $\overline{h}$ at the upper volatility bound and a BS-hedge for $\underline{h}$ at the lower volatility bound, c.f. Definition 3.1 of Branger and Mahayni (2006). It is easy to see that this strategy is indeed a superhedge for h.

The price bound corresponding to the *cheapest tractable robust hedge* is denoted by u^{Trac}, and the optimal choice of the two payoffs is denoted by $(\bar{h}^*, \underline{h}^*)$. In the special case where the claim to be hedged is convex (concave), the ALP–hedge is given by a Black–Scholes hedge at the upper (lower) volatility bound and is thus tractable. It then holds that

$$u^{\mathrm{Trac}}(t,x;\bar{h}) = u^{\mathrm{ALP}}(t,x;\bar{h}), \quad u^{\mathrm{Trac}}(t,x;\underline{h}) = u^{\mathrm{ALP}}(t,x;\underline{h}).$$

For the general case, it immediately follows that the initial investment which is needed to achieve a tractable robust hedge is at least as high as the lowest upper price bound, i.e.

$$u^{\mathrm{Trac}}(t,x;h) = u(t,x;\bar{h}^*) + u(t,x;\underline{h}^*) \geq u(t,x;\bar{h}) + u(t,x;\underline{h}) \geq u^{\mathrm{ALP}}(t,x;h). \tag{4}$$

To motivate the use of tractable strategies, note that the true model is not known. In such a situation, the use of BS–strategies is certainly the most simple choice. However, these strategies provide a superhedge only for convex or concave claims, but not for mixed payoffs. This problem is solved by the use of tractable strategies, without foregoing the simplicity of BS–strategies.

Branger and Mahayni (2006) determine the optimal tractable robust hedge. They show that it follows from the optimal decomposition of an optimal dominating payoff. The optimal decomposition of a payoff is given in their Proposition 3.3, and Proposition 3.6 characterizes the optimal dominating payoff. In particular, they show that it might be optimal to dominate the original payoff in case of tractable hedging. The basic intuition for this result is that the increase in initial capital due to dominating the payoff may more than be offset by the reduced overall curvature of the dominating payoff. For a more detailed explanation and some examples, we refer the reader to Branger and Mahayni (2006).

We conclude this section by reviewing some basic and helpful properties of the price bounds which hold both for the ALP–hedge and for the tractable hedge. Note that $-u^w(t,x;-h)$ gives the highest lower price bound for the claim h.

Lemma 2.1 (Properties of price bounds). *For $w \in \{Trac,\ ALP\}$, the lowest upper price bounds $u^w(t,x;f_1)$ and $u^w(t,x;f_2)$ at time $t \in [0,T[$ and state x as a function of the payoffs f_1 and f_2 at maturity T satisfy the following conditions:*

(i) Monotonicity: $f_1(y) \geq f_2(y)\ \forall y \geq 0 \Rightarrow u^w(t,x;f_1) \geq u^w(t,x;f_2)\ \forall x \geq 0$

(ii) Subadditivity: $u^w(t,x;f_1) + u^w(t,x;f_2) \geq u^w(t,x;f_1+f_2)\ \forall x \geq 0$

(iii) $f_1,\ f_2$ *convex or* $f_1,\ f_2$ *concave* $\Rightarrow u^w(t,x;f_1)+u^w(t,x;f_2) = u^w(t,x;f_1+f_2)\ \forall x \geq 0$

(iv) $f_2(y) = a + by\ \forall y \geq 0 \Rightarrow u^w(t,x;f_1) + u^w(t,x;f_2) = u^w(t,x;f_1+f_2)\ \forall x \geq 0$

(v) $u^w(t,x;f_2) \geq -u^w(t,x;-f_2)\ \forall x \geq 0.$

Proof: The proof of the case $w =$ ALP is given in Branger and Mahayni (2005). The case $w =$ Trac can be proved along the same lines. □

The price bounds depend monotonically on the volatility bounds, as shown in the next lemma:

Lemma 2.2 (Monotonicity of price bounds in volatility bounds). *For $w \in \{Trac,\ ALP\}$, the lowest upper price bound $u^w(t,x;h,\sigma_{min},\sigma_{max})$ at time $t \in [0,T[$ and in state x for the payoff h at maturity T is monotonically increasing in the upper volatility bound σ_{max} and monotonically decreasing in the lower volatility bound σ_{min}.*

Proof: To prove the monotonicity in the upper volatility bound, note that $\sigma_{\min} \leq \sigma_1 \leq \sigma_2$ implies $\mathcal{Q}_1^* \subseteq \mathcal{Q}_2^*$. The case $w =$ ALP is then obvious, since

$$\sup_{Q \in \mathcal{Q}_1^*} \mathbb{E}^Q[h(X_T)|\mathcal{F}_t] \ \leq \ \sup_{Q \in \mathcal{Q}_2^*} \mathbb{E}^Q[h(X_T)|\mathcal{F}_t].$$

Furthermore, it holds that

$$\begin{aligned} & Inf_{\overline{h},\underline{h}} \left[\sup_{Q \in \mathcal{Q}_1^*} \mathbb{E}^Q[\overline{h}(X_T)|\mathcal{F}_t] + \sup_{Q \in \mathcal{Q}_1^*} \mathbb{E}^Q[\underline{h}(X_T)|\mathcal{F}_t] \right] \\ \leq \ & Inf_{\overline{h},\underline{h}} \left[\sup_{Q \in \mathcal{Q}_2^*} \mathbb{E}^Q[\overline{h}(X_T)|\mathcal{F}_t] + \sup_{Q \in \mathcal{Q}_2^*} \mathbb{E}^Q[\underline{h}(X_T)|\mathcal{F}_t] \right]. \end{aligned}$$

which proves the case $w =$ Trac. The proof for the monotonicity in the lower volatility bound is analogous. □

3. Additional Hedge Instrument

The claim to be hedged is given by its payoff–function h. Besides the stock X and the bond B, the investor can now also use an additional claim g for hedging. Throughout the following, we assume that g is convex, and we exclude the trivial case of an affine g.[7] The (market) price M of g satisfies for all $t \in [0,T]$ the no arbitrage condition $M_t(g) \in]-u(t,S_t;-g), u(t,S_t;g)[$. We exclude the bounds, since in this case we could infer from the market price of g that the true volatility is equal to either the upper or the lower volatility bound. Then, we would no longer be in the uncertain volatility model.

3.1. **Hedge With Additional Hedge Instrument.** Along the lines of Branger and Mahayni (2006) we define a tractable robust hedge with an additional hedge instrument as follows:

Definition 3.1 (Tractable Robust Hedge with Additional Hedge Instrument). *A tractable robust hedge with additional instrument g for h is represented by a triple $(\overline{h}, \underline{h}, \phi)$ where $\overline{h}$ is a convex payoff, $\underline{h}$ is a concave payoff, and where*

$$\overline{h}(x) + \underline{h}(x) + \phi g(x) \quad \geq \quad h(x) \quad \forall x \geq 0.$$

The tractable robust hedge is given by ϕ static positions in g, a BS-hedge for $\overline{h}$ at the upper volatility bound, and a BS-hedge for $\underline{h}$ at the lower volatility bound.

We are interested in the optimal position in the additional claim, and in the reduction in the initial capital due to the availability of the additional instrument g. For $w \in \{\text{Trac}, \text{ALP}\}$, the lowest upper price bounds u^w with additional hedging instrument are[8]

$$u^w(t,S_t;h|g) \quad = \quad \text{Inf}_\phi \left\{ \phi M_t(g) + u^w\left(t,S_t;h-\phi g\right) \right\}.$$

The optimal position in the additional claim is given in the next definition:

Definition 3.2 (Optimal Position in Additional Hedge Instrument). *For $w \in \{Trac,\ ALP\}$, the optimal position in g for hedging h is denoted by ϕ^*_w and solves the minimization problem*

$$\phi^*_w := argmin_\phi \left\{ \phi M_t(g) + u^w\left(t,x;h-\phi g\right) \right\}.$$

For $w = Trac$, $(\bar{h}^*_{Trac}, \underline{h}^*_{Trac})$ represents the optimal tractable robust hedge for the modified payoff $h - \phi^*_{Trac} g$, and the optimal tractable robust hedge with additional instrument for the payoff h is represented by $(\bar{h}^*_{Trac}, \underline{h}^*_{Trac}, \phi^*_{Trac})$.

We are interested in the positions which give the cheapest tractable hedge, i.e. we want to characterize the triple $(\bar{h}^*_{\text{Trac}}, \underline{h}^*_{\text{Trac}}, \phi^*_{\text{Trac}})$ which is defined in Definition 3.2 . To simplify the notation, we only write $(\bar{h}^*, \underline{h}^*, \phi^*)$ instead of $(\bar{h}^*_{\text{Trac}}, \underline{h}^*_{\text{Trac}}, \phi^*_{\text{Trac}})$ in the following.

[7]Notice that a concave claim is simply given by a short position in a convex claim. We do not include mixed claims.

[8]For the unconstrained case $w = ALP$, we also refer to Avellaneda and Parás (1996).

3.2. **Optimal Position in the Additional Instrument.** First, we derive some properties of the optimal position in the additional instrument. To get the intuition, note that a positive ϕ implies that the hedger buys convexity at a market price which is lower than the price implied by the upper volatility bound. If $\phi < 0$, he sells convexity at the market price, which is cheaper than selling it at the lower volatility bound.

The optimal position in the additional claim will depend on the price of this claim. The next lemma shows that, in line with intuition, the position is the larger the cheaper the claim is:

LEMMA 3.3 (Monotonicity of optimal static position in market price). *For $w \in \{Trac,\ ALP\}$, the optimal position ϕ_w^* in the additional instrument g is monotonically decreasing in the market price $M_t(g)$.*

PROOF: For $i = 1, 2$, let $M_{t,i}(g)$ be the market price of the claim g and let $\phi_{w,i}^*$ be the optimal static position in the instrument g. From the optimality of $\phi_{w,i}^*$, we first get

$$\begin{aligned}
\phi_{w,1}^* M_{t,1}(g) + u^w\left(t, x; h - \phi_{w,1}^* g\right) &\leq \phi_{w,2}^* M_{t,1}(g) + u^w\left(t, x; h - \phi_{w,2}^* g\right) \\
\phi_{w,2}^* M_{t,2}(g) + u^w\left(t, x; h - \phi_{w,2}^* g\right) &\leq \phi_{w,1}^* M_{t,2}(g) + u^w\left(t, x; h - \phi_{w,1}^* g\right).
\end{aligned}$$

Combining these equations gives

$$\left(\phi_{w,1}^* - \phi_{w,2}^*\right) M_{t,1}(g) + u^w\left(t, x; h - \phi_{w,1}^* g\right) \leq \left(\phi_{w,1}^* - \phi_{w,2}^*\right) M_{t,2}(g) + u^w\left(t, x; h - \phi_{w,1}^* g\right)$$

which implies

$$\left(\phi_{w,1}^* - \phi_{w,2}^*\right)\left(M_{t,1}(g) - M_{t,2}(g)\right) \leq 0.$$

$M_{t,1}(g) < M_{t,2}(g)$ thus implies $\phi_{w,1}^* \geq \phi_{w,2}^*$, so that the optimal static position in the claim is decreasing in its market price. □

This lemma establishes a relation between the position in the additional claim and the price of buying or selling convexity at the market price. We conjecture that there is a similar relation between the optimal position in the additional claim and the price of superhedging convexity and concavity. The optimal position in the convex claim g should increase if hedging convexity is becoming more expensive (i.e. if the upper volatility bound increases), and it should decrease if hedging concavity is becoming more expensive (i.e. if the lower volatility bound decreases):

LEMMA 3.4 (Monotonicity of optimal static position in volatility bounds). *For h, g convex and $w \in \{Trac,\ ALP\}$ it holds that ϕ_w^* is monotonically increasing (decreasing) in σ_{max} (σ_{min}).*

Proof: *Still to come*

Furthermore, if the investor faces a situation where he can either hedge a convex claim at the upper volatility bound or buy it at the market price, he will of course choose the cheaper alternative and buy it at the market price. This allows us to derive upper and lower bounds on the optimal position in the additional claim:

Proposition 3.5 (Bounds for optimal static position). *Let* $l := \sup\{\phi | h - \phi g \textit{ is convex}\}$ *and* $u := \inf\{\phi | h - \phi g \textit{ is concave}\}$. *For* $w \in \{Trac,\ ALP\}$ *it holds that* $\phi^*_w \in [l, u]$.

Proof: The proof is given in Appendix A.1

The above proposition is very intuitive. Consider the lower boundary $l \leq \phi^*$, e.g. and start with a position of l in the additional claim. Since the modified payoff $h - lg$ is convex, the investor might want to buy additional convexity at the market price (choosing $\phi > l$). However, it is never optimal to sell additional convexity (choosing $\phi < l$) and make the modified claim even more convex. In the latter case, the investor would actually sell convexity at the market price and (re-)buy it at the higher price implied by the upper volatility bound, which just increases the initial capital needed. Generalizing this idea, one can argue intuitively that the position in ϕ should be chosen such as to minimize (in a sense still to be made clear) the overall curvature of the remaining modified claim.

Recall that we assume that the additional hedging instrument is given by a convex claim, i.e. an option. A long position ϕ^* implies that the investor wants to buy convexity at the market price and thus cares more about hedging the convexity risk at the upper volatility bound than he cares about hedging the concavity risk at the lower volatility bound. ϕ^* can thus be interpreted as a measure for the convexity risk of the claim. If h is a mixed payoff which is neither convex nor concave, then the interesting question is of course whether the optimal solution is given by buying or selling convexity, that is whether the convexity or the concavity of the claim dominates. We will come back to this question when we study the example of a bullish vertical spread. Second, it is interesting to see whether the sign of the optimal position in g depends on whether we consider a tractable hedge or an ALP–hedge. From Proposition 3.5, we know that if the original claim is convex, then both hedges imply a long position (or no position) in the additional claim, and if the original claim is concave, they both imply a short position (or no position). Again, things are more complicated for mixed payoffs, as we will show in the example of the bullish vertical spread again.

3.3. **Optimal Tractable Hedge of the Remaining Payoff with AI.** Up to now, we have considered the optimal position in the additional instrument. Now we turn to the remaining payoff $h - \phi g$ which still has to be hedged at the upper and lower volatility bound.

Proposition 3.6 (Structure of Optimal Tractable Robust Hedge). *For the cheapest tractable robust hedge* $\left(\bar{h}^*, \underline{h}^*, \phi^*\right)$ *with additional instrument* g*, it holds that the cheapest tractable robust hedge* $\left(\bar{h}^*, \underline{h}^*\right)$ *of the modified payoff* $h - \phi^* g$ *must not include a long or a short position in* g*, i.e. there does not exist any* $\alpha > 0$ *such that* $\bar{h}^* - \alpha g$ *is convex or* $\underline{h}^* + \alpha g$ *is concave.*

Proof: The proof is given in Appendix A.2.

The intuition is similar as for Proposition 3.5. Consider the payoff $\bar{h}^*$. The investor hedges this payoff at the upper volatility bound. If there is an $\alpha > 0$ such that $\bar{h}^* - \alpha g$ is convex, then he can buy the convexity of α positions in the additional instrument at the market price, and the payoff he has to hedge at the (more expensive) upper volatility bound is reduced to $\bar{h}^* - \alpha g$. Put differently, hedging at the upper volatility bound can only be optimal if buying g does not provide an easy alternative. A similar argument of course holds for the concave payoff $\underline{h}^*$.

4. Illustrative Examples

To illustrate the use of an additional hedge instrument, we consider two example. In the first example, we deal with the simplest case where both the claim to be hedged and the additional instrument are call options. It turns out that the optimal tractable hedge can be given in closed form and is interpretable by a simple convexity argument. Besides, the example gives an easy motivation for more general scenarios. In the second example, we consider a bullish vertical spread and thus deal with a claim that is neither convex nor concave.

4.1. **Hedging call options.**

The payoff function of a plain vanilla call option with maturity T and strike K is denoted by c_K, i.e. we have

$$c_K := (x - K)^+ .$$

We consider the cases $h = c_K$ and $h = -c_K$, and we will show that there are indeed distinctive differences between superhedging the payoff of a call long and a call short. The additional claim is a call option, too, i.e. $g = c_{K_M}$. We assume that the maturity of all options which are considered is T, but that the strikes are different, i.e. $K \neq K_M$.

Before we give the solution for the cheapest tractable robust hedge, we define some special strike prices which will turn out to be useful later on:

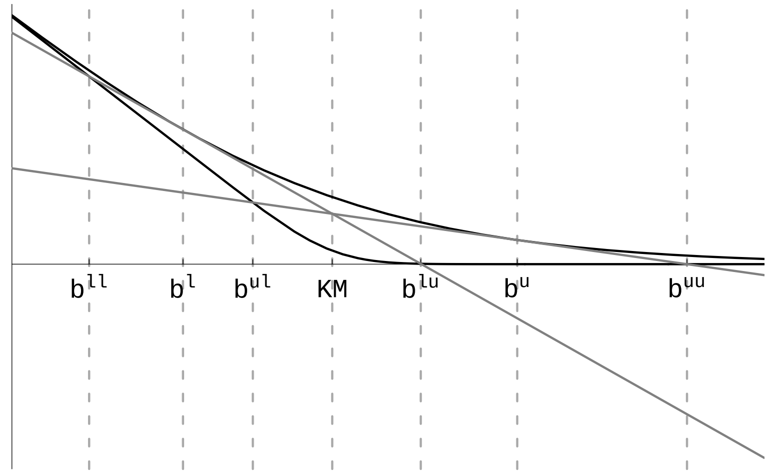

FIGURE 1. The figure illustrates the definition of the strikes in Lemma 4.1. The tangents to the upper price bound at the strikes b^l and b^u go through the market price of the additional claim. The intersections of these tangents with the lower price bound then define the strikes b^{ll}, b^{lu}, b^{ul} and b^{uu}.

LEMMA 4.1. *Depending on the upper and lower volatility bound and the strike and market price of the additional instrument, we can define the following six special strike prices:*

(a) The equation

$$\frac{\partial u(t,x;c_b)}{\partial b} = \frac{u(t,x;c_b) - M_t(c_{K_M})}{b - K_M} \tag{5}$$

has two solutions $b = b^l$ and $b = b^u$ with $b^l < K_M < b^u$. In the limiting case $\sigma_{max} \to \infty$, these solutions converge to $\lim_{\sigma_{max}\to\infty} b^l = 0,\ \lim_{\sigma_{max}\to\infty} b^u = \infty$.

(b) For $i = l, u$, the equation

$$\frac{-u(t,x;-c_{\tilde{b}}) - M_t(c_{K_M})}{\tilde{b} - K_M} = \frac{u(t,x;c_{b^i}) - M_t(c_{K_M})}{b^i - K_M}. \tag{6}$$

has two solutions $\tilde{b} = b^{il}$ and $\tilde{b} = b^{iu}$ with $b^{il} < K_M < b^{iu}$.

PROOF: The proof is given in Appendix B.1.

In Lemma 4.1, the upper and lower price bounds $u(.;c_b)$ and $-u(.;-c_b)$ of a European call are considered as a function of the strike price b. The solution of Equation (5) is a strike such that the tangent line to the upper price bound goes through the price of the traded call with strike K_M, c.f. Figure 1. According to Lemma 4.1, Equation (5) has two solutions b^l and b^u with $b^l < K_M < b^u$. Furthermore, each tangent line intersects the lower price bound twice, and this gives the strikes $b^{ll} < K_M < b^{lu}$ (for the tangent at $b = b^l$) and $b^{ul} < K_M < b^{uu}$ (for the tangent at $b = b^u$) which solve Equation (6).

In the following, we represent the cheapest tractable hedge along the lines of Definition 3.2, i.e. by its optimal static position ϕ^* in the additional claim g and the tuple $\left(\bar{h}^*, \underline{h}^*\right)$ which gives the optimal tractable robust hedge for the remaining payoff $h - \phi^* g$. In particular, the position in the underlying is given by the delta hedge for $\bar{h}^*$ at the upper volatility bound and $\underline{h}^*$ at the lower volatility bound.

Proposition 4.2 (Cheapest Tractable Robust Hedge with AI – Call Long). *The cheapest tractable hedge of a call with strike K is given by*

$$
\begin{aligned}
\phi^* &= \frac{K - b^l}{K_M - b^l} 1_{\left\{b^l < K < K_M\right\}} + \frac{K - b^u}{K_M - b^u} 1_{\{K_M < K < b^u\}} \\
\bar{h}^* &= (1 - \phi^*)\, c_{b^l}\, 1_{\left\{b^l < K < K_M\right\}} + (1 - \phi^*)\, c_{b^u}\, 1_{\{K_M < K < b^u\}} + c_K\, 1_{\left\{K \notin [b^l, b^u]\right\}} \\
\underline{h}^* &= 0
\end{aligned}
$$

where b^l and b^u are defined as in Lemma 4.1. It holds that $\phi^* \in [0, 1]$.

Proof: From Proposition 3.5, it follows that $\phi \geq 0$. The static position ϕ in the claim $g = c_M$ results in the modified payoff

$$
h(x) - \phi g(x) = (x - K)^+ - \phi(x - K_M)^+. \tag{7}
$$

This payoff is structurally equal to the payoff of a vertical spread. In the special case $\phi = 1$, we get a bearish (bullish) vertical spread for $K > K_M$ ($K < K_M$).

Branger and Mahayni (2005) show that the cheapest tractable hedge is given by the optimal decomposition of the optimal dominating payoff. Applying Proposition 3.6 of Branger and Mahayni (2005) to the modified payoff (7) shows that the optimal dominating portfolio consists of a short position in a call with strike K_M and a long position in a call with strike b.[9] For $K_M < K$ ($K_M > K$) we have $b > K$ ($b < K$). Let α_i denote the number of calls with strike i ($i = b, K_M$). The portfolio of $\phi + \alpha_{K_M}$ options with strike K_M and α_b calls with strike b has to dominate the original call with strike K, and it must not be possible to reduce the position in the options any further. This gives the following conditions

$$
\begin{aligned}
&(i) \quad \alpha_{K_M} + \phi + \alpha_b = 1 \\
&(ii) \quad (\alpha_{K_M} + \phi)(b - K_M) = b - K
\end{aligned}
$$

Furthermore, we must have $\alpha_b \geq 0$. Proposition 3.6 immediately implies $\alpha_{K_M} = 0$, i.e. the hedge of the modified payoff must not include a superhedge of the additional hedging instrument. Summing up, the structure of the optimal hedge is given by

$$
\alpha_{K_M} = 0, \quad \alpha_b = 1 - \phi, \quad \phi = \frac{K - b}{K_M - b}
$$

[9] Notice that these option positions are superhedged dynamically.

Lower and Upper Price Bounds for Call with AI

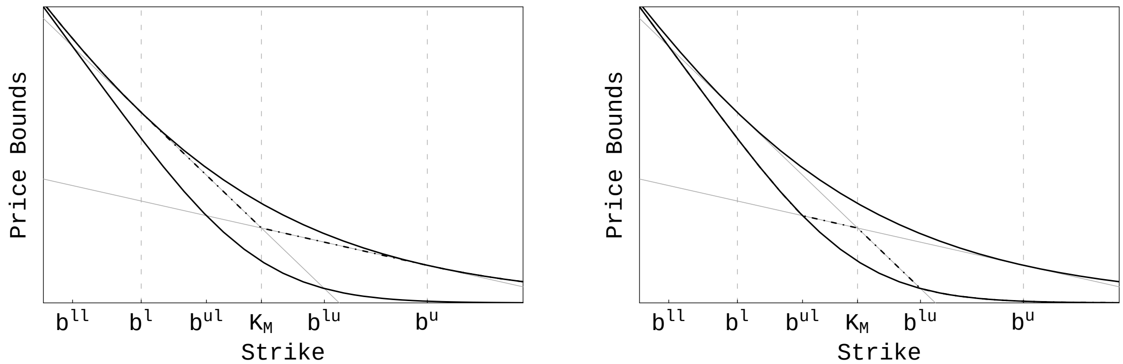

FIGURE 2. The figures show the upper and lower price bound (solid black lines) for a call as a function of the strike, as well as the tangents (solid grey lines) to the upper price bound that pass through the price of the traded call. If an additional call with strike K_M is available, then the price bounds can be tightened. The left figure shows the new upper price bound (dash-dotted line), which is lower if $b^l < K < b^u$. The right figure shows the new lower price bound(dash-dotted line), which is larger if $b^{ul} < K < b^{lu}$.

It can easily be seen that $\alpha_b \in [0,1]$ and $\phi \in [0,1]$. The position α_b in the option with strike b is superhedged dynamically, i.e. a long position in this call is delta–hedged at the upper volatility bound. In contrast, the call with strike K_M is available at the market. The (total) initial investment for the hedging positions is given by

$$\left(1 - \frac{K-b}{K_M - b}\right) u(t,x;c_b) + \frac{K-b}{K_M-b} M_t(c_{K_M}). \tag{8}$$

In the last step, we have to find the optimal b. From the first order condition and Lemma 4.1, we get $b = b^l$ or $b = b^u$. For $K_M < K < b^u$, we thus set $b = b^u$, and for $b^l < K < K_M$, we set $b = b^l$. Otherwise, if $K \notin [b^l, b^u]$, then the optimal choice of b is not viable, i.e. the condition $\phi \in [0,1]$ is violated. Then, we get the boundary solution $b = K$, and it is optimal just to superhedge the call with strike K at the upper volatility bound. □

The upper price bound for tractable hedging and if an additional call is available is illustrated in the left panel of Figure 2. If the strike of the claim to be hedged is between b^l and b^u, then the new price bound is indeed lower than the old one. Following Proposition 4.2, the call with strike K is dominated by a portfolio of two calls with a lower and a higher strike price. If all three calls were priced and hedged at the upper volatility bound, then the initial capital needed for the dominating payoff would of course be larger. However, the call with strike K_M can be bought at a lower market price instead of being hedged at the upper volatility bound, which reduces the initial capital needed for the convex combination. It then depends on the tradeoff between these two effects whether the additional call can be used to lower the upper price bound. If K is between b^l and b^u, the weight of the traded call in the hedge portfolio is large enough for the second effect to be more important, and the hedge becomes cheaper.

It is also interesting to look at the limiting hedge when the upper volatility bound goes to infinity or when the strike of the additional option converges towards the strike of the option to be hedged.

Corollary 4.3. *[Limits for the Cheapest Tractable Hedge with AI – Call Long]*

(i) In an uncertain volatility model where $\sigma_{max} \to \infty$, the optimal tractable robust hedge for c_K converges to a static portfolio:
(a) for $K_M > K$: portfolio of the stock $(1-\phi^*)$ and the additional call with $\phi^* = \frac{K}{K_M}$
(b) for $K_M < K$: position in the additional call with $\phi^* = 1$

(ii) The optimal static position is the higher the closer the additional strike is to the strike of the option to be hedged, i.e. $\lim_{K_M \to K} \phi^* = 1$.

Proof: The above corollary is an immediate consequence of Lemma 4.1 and Proposition 4.2. □

Note that the above corollary is also true in the unconstrained case which is considered in Avellaneda and Parás (1996), i.e. for the limiting cases $\sigma_{\max} \to \infty$ and $K_M \to K$ we have $\phi^*_{\text{Trac}} = \phi^*_{\text{ALP}}$. In general, the optimal static position of the constrained scenario differs from the one in the unconstrained case. This and the above results are illustrated by a numerical example.

Along the lines of Avellaneda and Parás (1996), we consider the problem of hedging OTC options with different strikes using ATM options. For simplicity, we set $K_M = x = 100$. We use the same parameters as ALP. The volatility band is defined by $\sigma_{min} = 0.08$ and $\sigma_{max} = 0.32$. The risk-free interest rate is set to $r = 0.06$, and time to maturity is equal to half a year. Finally, the implied volatility of the ATM option available for hedging is $\sigma_{imp} = 0.16$. The call options to be hedged have moneyness (defined as strike price over stock price) ranging from 0.85 to 1.15.

Table 1 shows the results both for the ALP–hedge and the tractable hedge with and without the additional instrument. Without the additional instrument, both the restricted and the unrestricted hedge coincide, and the optimal strategy is just to hedge the call at the upper volatility bound. If the additional ATM–call is available, the initial capital is lower in the unrestricted ALP–case than for the restricted tractable hedge. With an additional instrument is used, the remaining payoff is mixed, which implies that the

$\frac{K}{K_M} = \frac{K}{100}$	0.85	0.90	0.95	1.00	1.05	1.10	1.15
$u^{\text{ALP}}(t,x;c_K)$	19.7623	16.2269	13.1063	10.4343	8.1739	6.3223	4.82708
$u_x^{\text{ALP}}(t,x;c_K)$	0.8324	0.7616	0.6818	0.5971	0.5120	0.4303	0.3550
$u^{\text{ALP}}(t,x;c_K\|c_{K_M})$	18.6884	14.3800	10.1787	6.0773	4.9385	3.9769	3.1640
$u_x^{\text{ALP}}(t,x;c_K\|c_{K_M})$	0.5936	0.4122	0.2092	0.0000	−0.0776	−0.0902	−0.0784
ϕ^*_{ALP}	0.4150	0.6038	0.8074	1.0000	0.9444	0.8172	0.6727
$u^{\text{Trac}}(t,x;c_K)$	19.7623	16.2269	13.1063	10.4343	8.1739	6.3223	4.82708
$u_x^{\text{Trac}}(t,x;c_K)$	0.8324	0.7616	0.6818	0.5971	0.5120	0.4303	0.3550
$u^{\text{Trac}}(t,x;c_K\|c_{K_M})$	19.1799	14.8124	10.4449	6.0773	5.3963	4.7153	4.03431
$u_x^{\text{Trac}}(t,x;c_K\|c_{K_M})$	0.5639	0.3759	0.1880	0.0000	0.0349	0.0698	0.1047
ϕ^*_{Trac}	0.3964	0.5976	0.7988	1.0000	0.8228	0.6455	0.4683

Table 1. Comparison of constrained and unconstrained solutions.

hedge of this remaining payoff is more expensive for the tractable strategy than for the ALP–strategy, as already argued in Equation (4). The investor thus profits less from the availability of the additional call when he is restricted to tractable trading strategies.

For all strikes, the investor takes a long position in the additional instrument both for the tractable hedge and for the ALP-hedge. Given that he wants to hedge an already convex payoff, this result is intuitive and also follows from Proposition 3.5. The position in the additional instrument is larger in case of the ALP–hedge. Intuitively, this can again be explained by noting that the remaining payoff is neither convex nor concave. This makes the tractable hedge more expensive, and the investor is thus more reluctant to add this concavity to the originally convex claims. This in turn implies that he takes a smaller position in the additional instrument in case of the tractable hedge.

In the next proposition, we give the results for superhedging a call short:

Proposition 4.4 (Cheapest Tractable Robust Hedge with AI – Call Short). *The cheapest tractable hedge of a call short with strike K is given by*

$$
\begin{aligned}
\phi^* &= -\frac{K-b^u}{K_M-b^u}\mathit{1}_{\{b^{ul}<K<K_M\}} - \frac{K-b^l}{K_M-b^l}\mathit{1}_{\{K_M<K<b^{lu}\}} \\
\bar{h}^* &= -\phi^*\left(\frac{K-K_M}{K-b^u}\, c_{b^u}\ \mathit{1}_{\{b^{ul}<K<K_M\}} + \frac{K-K_M}{K-b^l}\, c_{b^l}\ \mathit{1}_{\{K_M<K<b^{lu}\}}\right) \\
\underline{h}^* &= -c_K\ \mathit{1}_{\{K\notin[b^{ul},b^{lu}]\}}
\end{aligned}
$$

where b^l, b^{lu}, b^u, and b^{ul} are defined in Lemma 4.1. In particular, $\phi^* \in [-1,0]$.

Proof: The proof is given in Appendix B.2.

The result is illustrated in the right panel of Figure 2 which shows the lower price bound if an additional instrument is available. First, the payoff of the call short is dominated by the payoff of a short position in a call with strike K_M and a long position in a call with strike b^i. If the short position would be priced at the lower volatility bound and the long position at the upper volatility bound, then the resulting lower price bound would indeed be smaller than the old one. However, the short position in the traded call is traded at the market price, which increases the lower price bound. And again, if $b^{ul} < K < b^{lu}$, then the weight of the traded call in the dominating portfolio is large enough for the investor to profit from the additional call.

4.2. **Hedge of Bullish Vertical Spread.** In the next example, we consider hedging a bullish vertical spread with strikes $K_1 < K_2$, i.e. $h(x) = (x - K_1)^+ - (x - K_2)^+$. While the call was an example for a convex payoff, the bullish vertical spread is neither convex nor concave. This leads to two interesting questions. First, the tractable robust hedge and the ALP–hedge do not coincide even in the base case without an additional traded instrument, and the tractable robust hedge is more expensive than the ALP–hedge, i.e. $u^{\text{ALP}}(t, x, h) < u^{\text{Trac}}(t, x, h)$. It is then no longer obvious whether the additional instrument will reduce the initial capital by more in the unconstrained case or in the constrained case. Second, the optimal static position in the additional claim can be positive as well as negative. Thus, it is interesting to analyze under which conditions the investor will buy convexity at the market price, and under which conditions he will sell it.

For the case that $K_M < K_1 < K_2$, the representation of the optimal tractable hedge with an additional call is given in the following proposition. The remaining cases $K_1 < K_M < K_2$ and $K_1 < K_2 < K_M$ can be found in Appendix C.

Proposition 4.5 (Tractable Robust Hedge of Bullish Call Spread with Additional Instrument). *The cheapest tractable robust hedge with additional instrument c_{K_M} for a bullish vertical spread with payoff–function $h(x) = (x - K_1)^+ - (x - K_2)^+$ is for $K_M < K_1 < K_2$*

given by

$$
\begin{aligned}
\phi^* &= -\frac{K_2-K_1}{K_M-b^l}\mathit{1}_{\{K_2<b^{lu}\}}+\frac{K_2-K_1}{K_2-K_M}\mathit{1}_{\{b^{lu}<K_2<b^{uu}\}}+\frac{b^u-K_1}{b^u-K_M}\mathit{1}_{\{K_1<b^u\}}\mathit{1}_{\{b^{uu}<K_2\}}\\
\bar{h}^* &= \frac{K_2-K_1}{K_M-b^l}\,c_{b^l}\,\mathit{1}_{\{K_2<b^{lu}\}}\\
&\quad+\left(\frac{K_1-K_M}{b^u-K_M}\,c_{b^u}\,\mathit{1}_{\{K_1<b^u\}}+\frac{K_2-K_1}{K_2-a^*}\,c_{a^*}\,\mathit{1}_{\{b^u<K_1\}}\right)\mathit{1}_{\{b^{uu}<K_2\}}\\
\underline{h}^* &= -\frac{K_2-K_1}{K_2-K_M}\,c_{K_2}\,\mathit{1}_{\{b^{lu}<K_2<b^{uu}\}}\\
&\quad+\left(-c_{K_2}\,\mathit{1}_{\{K_1<b^u\}}-\frac{K_2-K_1}{K_2-a^*}\,c_{K_2}\,\mathit{1}_{\{b^u<K_1\}}\right)\mathit{1}_{\{b^{uu}<K_2\}}
\end{aligned}
$$

where $a^*=\min\{a^{cand},K_1\}$ *and* a^{cand} *solves the equation*

$$
\frac{\partial u(t,X_t;c_a)}{\partial a} = \frac{-u(t,X_t;-c_{K_2})-u(t,x;c_a)}{K_2-a}.
$$

b^l (b^{ll},b^{lu}) *and* b^u (b^{ul},b^{uu}) *are defined as in Lemma 4.1. In particular,* $\phi^*\in(-\infty,1]$.

Proof: First, consider the structure of the hedge. For a given ϕ, the optimal tractable hedge of the remaining payoff follows from Proposition 3.6 in Branger and Mahayni (2006). Furthermore, Proposition 3.6 in this paper states that this hedge must not include a hedge of the additional call c_{K_m} at the upper or lower volatility bound. The hedge for the remaining payoff is then given by a long position in an option with strike y, where $y<K_M$ for $\phi<0$ and $y<K_2$ for $\phi\geq 0$, and a short position in the option with strike K_2. For the positions α_y and α_{K_2} in the options, a similar argumentation as in the proof of Proposition 4.2 gives

$$
\begin{aligned}
\alpha_y &= \frac{K_2-K_1}{K_2-y}-\phi\frac{K_2-K_M}{K_2-y}\\
\alpha_{K_2} &= -\frac{K_2-K_1}{K_2-y}-\phi\frac{K_M-K^l}{K_2-y}
\end{aligned}
$$

where y and ϕ have to meet one of the following two conditions

$$
\begin{aligned}
y<K_M \quad &\text{and} \quad \phi\in\left[-\frac{K_2-K_1}{K_M-y},0\right]\\
y<K_2 \quad &\text{and} \quad \phi\in\left[\frac{y-K_1}{y-K_M}\,1_{\{K_1<y<K_2\}},\frac{K_2-K_1}{K_2-K_M}\right].
\end{aligned}
$$

Second, we have to find the optimal choice of y and ϕ. The initial investment for the hedge is

$$\begin{aligned} V_t(\phi, y) &:= \frac{K_2 - K_1}{K_2 - y}\big[u(t,x;c_y) - (-1)u(t,x;-c_{K_2})\big] + \phi f(y) \\ \text{where } f(y) &:= M_t(c_{K_M}) - \frac{K_2 - K_M}{K_2 - y}u(t,x;c_y) - \frac{K_M - y}{K_2 - y}(-1)u(t,x;-c_{K_2}). \end{aligned}$$

For fixed y, we denote the optimal choice of ϕ by $\phi(y)$. It is easily seen that $\phi(y)$ is equal to the lower boundary if $f(y) > 0$ and equal to the upper boundary if $f(y) \leq 0$:

$$\phi(y) = \begin{cases} -\frac{K_2-K_1}{K_M-y} 1_{\{y<K_M\}} + 0 \cdot 1_{\{K_M<y<K_1\}} + \frac{y-K_1}{y-K_M} 1_{\{K_1<y<K_2\}} & f(y) > 0 \\ \frac{K_2-K_1}{K_2-K_M} & f(y) \leq 0 \end{cases}$$

The optimal y then follows from the first order conditions, where we have to distinguish between the cases $f(y) \leq 0$ and $f(y) > 0$ with $y < K_M$, $K_M < y < K_1$, and $K_1 < y < K_2$. Note that a necessary condition for $f(y) > 0$ to hold is $K_2 < b^{lu}$ or $K_2 > b^{uu}$. This gives the following candidate choices for y:

- in the interval $[0, K_M]$: b^l
- in the interval $[K_M, K_1]$: $\max\{K_M, a^*\}$
- in the interval $[K_1, K_2]$: $\max\{K_1, \min\{b^u, K_2\}\} 1_{\{K_2 \notin (b^{lu}, b^{uu})\}} + K_2 1_{\{K_2 \in [b^{lu}, b^{uu}]\}}$

where we have already dropped some choices that are never optimal. Comparing the initial investments that result from these cases then gives Proposition 4.5. □

For the numerical example, we use the same parameters as in Section 4.1, i.e. we set $\sigma_{min} = 0.08$, $\sigma_{max} = 0.32$, and $r = 0.06$. The inital stock price is $x = 100$, and the additional instrument is an ATM-call with implied volatility $\sigma_{imp} = 0.16$. The time to maturity of the contracts is again equal to half a year. We set $K_2 = K_1 + 10$.

Figure 3 shows the initial capital needed for the ALP-hedge and for the tractable hedge as a function of K_1, where we consider both the case with and without an additional instrument. The results confirm that the tractable hedge is more expensive than the ALP-hedge, which is to be expected for a mixed payoff. For both types of hedges, the initial capital can be reduced when an initial instrument is available, where the reduction depends on the relation between the strikes of the straddle and the strike of the additional call. If the claim to be hedged and the additional call are too different, that is if the strikes of the straddle are too far away from the strike K_M, the reduction in initial capital is zero for the tractable hedge and goes to zero for the ALP-hedge. Furthermore, the reduction in initial capital for the tractable hedge is zero for $K_2 \approx b^{lu}$. In this case, the optimal tractable hedge without the additional instrument cannot be improved upon, as can also

Lowest Upper Price Bound for Bullish Vertical Straddle

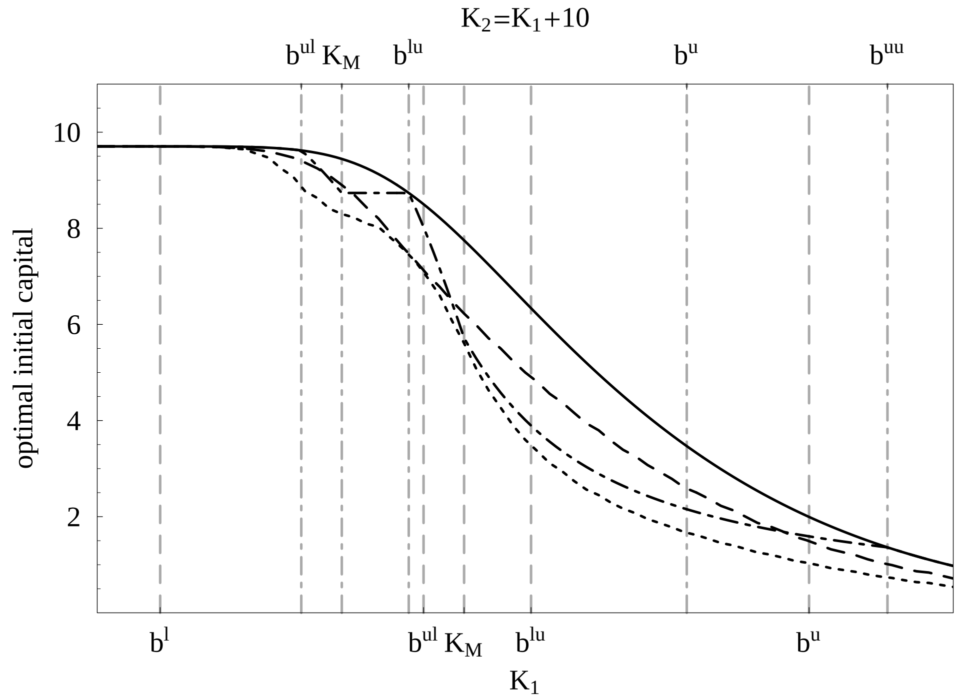

FIGURE 3. The figure shows the lowest upper price bound for a straddle with strikes K_1 (shown at the lower axis) and $K_2 = K_1 + 10$ (shown at the upper axis) as a function of K_1. We consider the ALP–hedge (dashed line), the ALP-hedge with a traded ATM-call (dotted line), the tractable hedge (solid line), and the tractable hedge with a traded ATM-call (dash-dotted line).

be shown analytically. For the other values of K_1 and K_2, the reduction in initial capital can be significant.

A comparison of the reduction in initial capital by the additional instrument shows that there is no ranking between the ALP–hedge and the tractable hedge. Again, it depends on the parameters for which hedge the reduction is larger. The most important result, however, is that the tractable hedge with an additional instrument can be significantly cheaper than the ALP–hedge without additional instrument. In our example, this holds true if the strike of the additional call is smaller than the two strikes of the straddle. Then, the disadvantage of the tractable hedge due to the restriction of possible trading strategies can thus be offset by using an additional hedge instrument. And while the tractable hedge can be given in (nearly) closed form up to the solution of some nonlinear equations to find the characteristic strikes in Lemma 4.1, the ALP–hedge needs to be determined in a tree by backward induction.

Optimal Position in Additional Call for Bullish Vertical Straddle

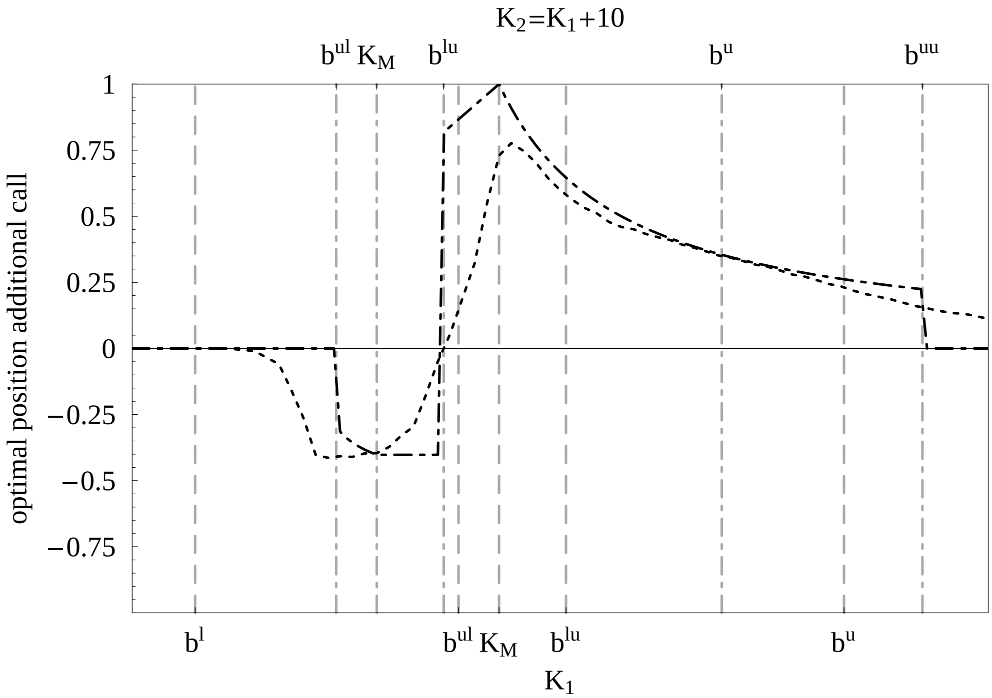

FIGURE 4. The figure shows the optimal position in a traded ATM-call for a straddle with strikes K_1 (shown at the lower axis) and $K_2 = K_1 + 10$ (shown at the upper axis) as a function of K_1. We consider the ALP-hedge with a traded ATM-call (dotted line) and the tractable hedge with a traded ATM-call (dash-dotted line).

The optimal position in the ATM-call is shown in Figure 4. The first question is whether the investor takes a long, a short, or no position in the additional call c_{K_M}, that is whether he buys or sells convexity at the market price. Intuitively, the answer should depend on whether the traded call is more similar to the convex part (call with strike K_1 long) or concave part (call with strike K_2 short) of the straddle. It should also depend on the current stock price in relation to the strikes of the straddle, which governs whether the straddle is more similar to a convex or to a concave claim at the moment. Consider the dependence on the strike prices first. As we can see from Propositions 4.5, C.1, and C.2, the distinctive parameter for the optimal tractable hedge is K_2. For $K_2 > b^{lu} > K_M$ and thus for a large strike price K_2, the investor takes a long position in the additional call, whereas he takes a short position for $K_2 < b^{lu}$. Surprisingly, the sign of the position in the additional call does not depend on K_1. For $K_1 > K_M$, the traded call with strike K_M is more similar to the call with strike K_1 (which would imply a long position) than to the call with strike K_2 (which would imply a short position). However, the investor still takes a short position in the additional call if $K_2 > b^{lu}$. Second, consider the dependence of the optimal position in the additional call on the current stock price. This dependence is indirect in that the optimal tractable hedge depends on the characteristic strikes from Lemma 4.1 which in turn depend on the current stock price. b^{lu} increases in the stock price, and there will be some critical stock price where the investor switches from a long position in the additional call (for stock prices below this level) to a short position (for stock prices above this level). This is in line with intuition: for low stock prices, the straddle is more similar to a convex claim (which implies a long position in the additional call), while it is more similar to a concave claim for large stock prices (which implies a short position).

In our example, the sign of the position in the additional call coincides for the tractable hedge and the ALP-hedge. The size of the position in the tractable hedge, however, can (in absolute terms) be both smaller and larger than the position in the ALP-hedge. The additional position is zero for the tractable hedge if the strike of the additional call is too different from the strikes of the straddle, and it goes to zero for the ALP-hedge if the difference between the strike of the call and the strikes of the straddle increases. Furthermore, note that the optimal position in the traded call is a continuous function of the strike K_1 for the ALP–hedge, while it jumps for the tractable hedge. At the discontinuity points, which are again defined in terms of K_2, there are two different strategies which are both optimal.

The cases $K_1 = K_M$ and $K_2 = K_M$ are special in that one of the component calls is already traded at the market. Intuitively, one would expect the investor to buy or sell the traded call and to hedge the remaining call with strike K_1 or K_2 at the upper or lower volatility bound. In our example, this intuition gives the right answer for $K_2 = K_M$, but not for $K_1 = K_M$, and it can be seen from Propositions 4.5, C.1, and C.2 that it holds only under some additional conditions. If $K_1 = K_M$, a long position in the traded call and a hedge of the call with strike K_2 at the lower volatility bound is optimal only if K_2 is large enough. If $K_2 = K_M$, a short position in the traded call and a hedge of the call with strike K_1 at the upper volatility bound is optimal only if K_1 is small enough.

Finally, we compare the optimal hedge for the straddle to the sum of the optimal hedges for the component payoffs. If no additional instrument is available, the sum of the ALP-hedges is given by hedging the call long at the upper volatility bound and the call short at the lower volatility bound. It is well known that this strategy is often prohibitively expensive compared to the optimal ALP-hedge for the portfolio. The same holds true if an additional instrument is used. For the tractable hedge, on the other hand, there are situations where the sum of the optimal hedges for the components is optimal for the portfolio, too. Without an additional instrument, this holds true if K_1 is small enough (for details, see **?**). With an additional instrument, this case may also happen. Assume e.g. $b^l < K_1 < K_M < K_2 < b^{lu}$. Adding up the optimal tractable hedges for the call long from Proposition 4.2 and the call short from Proposition 4.4 gives the optimal tractable hedge for the straddle from Proposition C.1 in Appendix C. However, there are other parameter scenarios where the sum of the hedges is more expensive than the hedge of the portfolio, and where the investor thus profits from a portfolio effect.

5. CONCLUSION

In this paper, we analyze the benefits of using an additional traded claim besides the stock and the money market account for hedging. We consider an uncertain volatility model with bounded volatility. The investor wants to implement a robust hedge, and he restricts the set of admissible strategies to tractable strategies, which can be written as the sum of Black–Scholes strategies. The additional instrument is supposed to be a call option.

The investor profits from the additional instrument. By trading the additional claim, he can buy and sell convexity at the market price instead of superhedging it at the volatility bounds, which lowers the initial investment. For the case of hedging a call option or a vertical spread, we derive closed form solutions for the optimal position in the additional instrument and for the initial capital. The optimal position in the additional call option can be interpreted as a measure for the convexity of the claim to be hedged. It is always positive in case of a call, but it may be both positive and negative for a bullish vertical spread. We also compare the results to the optimal hedge in Avellaneda et al. (1995) if the strategies are not restricted. The restriction leads to a more conservative position in the additional hedge instrument. Furthermore, the tractable hedge with an additional call can well be cheaper than the ALP–hedge without an additional instrument. At the same time, the optimal tractable hedge is much easier to implement and may be considered as the more intuitive hedging strategy.

Further research could proceed along several lines. First, the uncertain volatility model is one way to capture model risk, and the hedging strategies considered in this paper are robust with respect to this model class. It seems natural to extend the concept of robust hedging to other model classes, too, and to consider the use of additional instruments in these model classes. Second, it would be interesting to analyze so-called semi-static strategies, which are e.g. considered in Carr and Wu (2002), and to combine the criteria of robust hedging and semi-static hedging.

Appendix A. Proofs ad Section 3

A.1. **Proof of Proposition 3.5.** g convex implies $\{\phi | h - \phi g \text{ is convex}\} =]-\infty, l]$ and $\{\phi | h - \phi g \text{ is concave}\} = [u, \infty[$. Thus, the definition of l and u is sensible, and it holds that $l \leq u$. To prove $\phi_w^* \in [l, u]$ it is enough to show that $\phi M_t(g) + u^w(t, S_t; h - \phi g)$ is decreasing in ϕ for $\phi \in]-\infty, l]$ and increasing for $\phi \in [u, \infty[$.

First, assume that $\phi_1 < \phi_2 \leq l$.

$$\begin{aligned} & \phi_1 M_t(g) + u^w(t, S_t; h - \phi_1 g) - [\phi_2 M_t(g) + u^w(t, S_t; h - \phi_2 g)] \\ = \; & (\phi_1 - \phi_2) M_t(g) + u^w(t, S_t; h - \phi_1 g) - u^w(t, S_t; h - \phi_2 g) \\ = \; & (\phi_1 - \phi_2) M_t(g) + u^w(t, S_t; (\phi_2 - \phi_1) g) \\ = \; & (\phi_2 - \phi_1)[-M_t(g) + u^w(t, S_t; g)] \\ \geq \; & 0 \end{aligned}$$

where we have used that $h - \phi_1 g$, $h - \phi_2 g$ and $(\phi_2 - \phi_1) g$ are convex so that the price bounds are additive (see Lemma 2.1). This proves that $\phi M_t(g) + u^w(t, S_t; h - \phi g)$ is indeed decreasing in ϕ on the interval $(-\infty, l]$.

In a similar way, we can show that $\phi M_t(g) + u^w(t, S_t; h - \phi g)$ is increasing in ϕ on the interval $[u, \infty[$. □

A.2. **Proof of Proposition 3.6.** Along the lines of Definition 3.2 we have

$$\begin{aligned} (\bar{h}^*, \underline{h}^*, \phi^*) \;=\; & \operatorname{argmin}_{(\bar{h}, \underline{h}, \phi)} \left\{ \phi M_t(g) + u^{\mathrm{BSB}}\left(t, x; \bar{h}\right) + u^{\mathrm{BSB}}\left(t, x; \underline{h}\right) \right\}. \\ & \text{s.t. } \overline{h}(x) + \underline{h}(x) + \phi\, g(x) \;\geq\; h(x) \text{ for all } x \geq 0. \end{aligned}$$

Assume that there is a constant $\alpha > 0$ such that $\bar{h}^* - \alpha g$ is convex. Consider the triple $(\bar{h}, \underline{h}, \phi)$ where $\bar{h} = \bar{h}^* - \alpha g$, $\underline{h} = \underline{h}^*$ and $\phi = \phi^* + \alpha$. Obviously this triple also represents a tractable robust hedge. With the no arbitrage condition $M_t(g) \in]-u(t, S_t; -g), u(t, S_t; g)[$, the convexity of $\bar{h}^* - \alpha g$, and the convexity of αg, it follows that

$$\begin{aligned} & \phi M_t(g) + u^{\mathrm{BSB}}\left(t, x; \bar{h}\right) + u^{\mathrm{BSB}}\left(t, x; \underline{h}\right) \\ = \; & (\phi^* + \alpha) M_t(g) + u^{\mathrm{BSB}}\left(t, x; \bar{h}^* - \alpha g\right) + u^{\mathrm{BSB}}\left(t, x; \underline{h}^*\right) \\ = \; & \phi^* M_t(g) + u^{\mathrm{BSB}}\left(t, x; \bar{h}^*\right) + u^{\mathrm{BSB}}\left(t, x; \underline{h}^*\right) + \alpha \left(M_t(g) - u^{\mathrm{BSB}}(t, x; g)\right) \\ < \; & \phi^* M_t(g) + u^{\mathrm{BSB}}\left(t, x; \bar{h}^*\right) + u^{\mathrm{BSB}}\left(t, x; \underline{h}^*\right). \end{aligned}$$

The above contradicts the optimality of the triple $(\bar{h}^*, \underline{h}^*, \phi^*)$. Analogous arguments prohibit the existence of $\alpha > 0$ such that $\underline{h}^* + \alpha g$ is concave. □

Appendix B. Proofs ad Section 4

B.1. **Proof of Lemma 4.1.** The upper price bound for a European call is given by the BS–price at the upper volatility bound, i.e.

$$\begin{aligned} u(t,x;c_b) &= x\mathcal{N}\left(d_1\left(t,x;K;\sigma_{\max}\right)\right) - K\mathcal{N}\left(d_2\left(t,x;K;\sigma_{\max}\right)\right) \\ \text{where } d_1\left(t,x;K;\sigma_{\max}\right) &:= \frac{\ln\left(\frac{x}{K}\right) + \frac{1}{2}\sigma_{\max}^2(T-t)}{\sigma_{\max}\sqrt{T-t}} \\ d_2\left(t,x;K;\sigma_{\max}\right) &:= d_1\left(t,x;K;\sigma_{\max}\right) - \sigma_{\max}\sqrt{T-t} \end{aligned}$$

where $\mathcal{N}$ denotes the cumulative distribution function of the normal distribution.

Equation (5) can be rewritten as

$$f(t,x;b) := u(t,x;c_b) + \frac{\partial u(t,x;c_b)}{\partial b}(K_M - b) = M_t(c_{K_M})$$

It can be shown that the function f is monotonically increasing in b for $0 \leq b < K_M$ and monotonically decreasing in b for $K_M < b$. Furthermore, it holds that

$$\begin{aligned} \lim_{b \to 0} f(t,x;b) &= x - K_M < M_t(c_{K_M}) \\ f(t,x;K_M) &= u(t,x;c_{K_M}) > M_t(c_{K_M}) \\ \lim_{b \to \infty} f(t,x;b) &= 0 < M_t(c_{K_M}). \end{aligned}$$

Thus, the equation $f(t,x;b) = M_t(c_{c_{K_M}})$ has two solutions b^l, b^u where $0 < b^l < K_M$ and $b^u > K_M$. Since the upper price bound increases in $\sigma_{\max}$, if furthermore holds that $\lim_{\sigma_{\max}\to\infty} b^l = 0$ and $\lim_{\sigma_{\max}\to\infty} b^u = \infty$.

Equation (6) can be rewritten as

$$f(\tilde{b}) := M_t(c_{K_M}) + \frac{u(t,x;c_{b^i}) - M_t(c_{K_M})}{b^i - K_M}\left(\tilde{b} - K_M\right) = -u(t,x;-c_{\tilde{b}}) \tag{9}$$

where $i = l, u$. The function f is a monotonically decreasing and affine function of $\tilde{b}$, while the lower price bound $-u(t,x;-c_{\tilde{b}})$ is a convex function of $\tilde{b}$. Thus, Equation (9) has at most two solutions. Since f is the tangent to the convex upper price bound for the call, it holds that

$$f(0) < u(t,x;c_0) = x = -u(t,x;-c_0).$$

Furthermore, we know that

$$\begin{aligned} f(K_M) &= M_t(c_{K_M}) > -u(t,x;-c_{K_M}) \\ \lim_{\tilde{b}\to\infty} f(t,x;\tilde{b}) &= -\infty < 0 = \lim_{\tilde{b}\to\infty} -u(t,x;-c_{\tilde{b}}). \end{aligned}$$

Thus, for each $i = l, u$, Equation (9) has two solutions b^{il}, b^{iu} where $0 < b^{il} < K_M$ and $b^{iu} > K_M$. $\square$

B.2. **Proof of Proposition 4.4.** From Proposition 3.5, it follows that $\phi^* \leq 0$. The modified payoff $\tilde{h}$ is

$$\begin{aligned}\tilde{h}(x) &= h(x) - \phi g(x) \\ &= -\phi\,(x - K_M)^+ - (x - K)^+,\end{aligned}$$

which is similar to a bullish (bearish) vertical spread for $K_M < K$ ($K_M > K$).

For the case $K_M < K$, we know from Branger and Mahayni (2006) that the optimal dominating payoff is given by a long position in a call with strike $b < K_M$ and a short position in the call with strike K. Using the notation of the proof before and analogous arguments, we have

$$\begin{aligned}(i) &\quad \alpha_b + \phi + \alpha_K = -1 \\ (ii) &\quad \alpha_b(K - b) + \phi(K - K_M) = 0\end{aligned}$$

and we also know that

$$\alpha_b \geq 0,\ \alpha_K \leq 0$$

where $b \leq K_M$. This immediately gives

$$\begin{aligned}(i) &\qquad \alpha_b = -\phi\frac{K - K_M}{K - b} \\ (ii) &\qquad \alpha_K = -1 - \phi\frac{K_M - b}{K - b}.\end{aligned}$$

Furthermore, it has to hold that

$$(iii) \qquad -\frac{K - b}{K_M - b} \leq \phi \leq 0.$$

In the case of $K_M > K$, analogous arguments give

$$\begin{aligned}(i) &\qquad \alpha_b = -\phi\frac{K - K_M}{K - b} \geq 0 \\ (ii) &\qquad \alpha_K = -1 - \phi\frac{K_M - b}{K - b} \leq 0 \\ (iii) &\qquad -\frac{K - b}{K_M - b} \leq \phi \leq 0\end{aligned}$$

where $b > K_M$.

The (total) initial investment of the hedging positions is[10]

$$\begin{aligned}
& -\alpha_K u(t,x;-c_K) + \alpha_b u(t,x;c_b) + \phi M_t(c_{K_M}) \\
= & \left(1+\phi\frac{K_M-b}{K-b}\right) u(t,x;-c_K) - \phi\frac{K-K_M}{K-b} u(t,x;c_b) + \phi M_t(c_{K_M}) \\
= & \; u(t,x;-c_K) - \phi\left(\frac{K-K_M}{K-b}u(t,x;c_b) + \frac{K_M-b}{K-b}[-u(t,x;-c_K)] - M_t(c_{K_M})\right).
\end{aligned}$$

In case the term in brackets is positive, the optimal choice is $\phi^* = 0$, and the additional hedge instrument is not used. In the opposite case, we get $\phi^* = -\frac{K-b}{K_M-b}$. Then, the (total) initial investment is

$$\frac{K_M-K}{b-K_M}u(t,x;c_b) - \frac{b-K}{b-K_M}M_t(c_{K_M}),$$

and we have to find the optimal b in the next step. From the first order condition and Lemma 4.1, we get $b = b^l$ or $b = b^u$. For $K_M < K$, the optimal choice is thus $b = b^l$, and for $K < K_M$, it is $b = b^u$. Finally, it remains to show that

$$\frac{K-K_M}{K-b^i}u(t,x;c_{b^i}) + \frac{K_M-b^i}{K-b^i}[-u(t,x;-c_K)] - M_t(c_{K_M}) < 0 \iff K \in]b^{ul}, b^{lu}[.$$

Appendix C. ad Proposition 4.5

Proposition C.1 (Tractable Robust Hedge of Bullish Call Spread with Additional Instrument (ii)). *The cheapest tractable hedge with additional instrument c_{K_M} of a claim with payoff–function $h(x) = (x-K_1)^+ - (x-K_2)^+$ ($K_1 < K_M < K_2$) is given by*

$$\begin{aligned}
\phi^* &= \left[-\frac{K_2-K_1)}{K_M-b^l}1_{\{K_1>b^l\}} - \frac{K_2-b^l}{K_M-b^l}1_{\{K_1<b^l\}}\right]1_{\{K_2<b^{lu}\}} + \frac{K_1-b^l}{K_M-b^l}1_{\{K_2>b^{lu}\}}1_{\{K_1>b^l\}} \\
\bar{h}^* &= \left[\frac{K_2-K_1}{K_M-b^l}c_{b^l}1_{\{K_1>b^l\}} + \left(c_{K_1} + \frac{K_2-K_M}{K_M-b^l}c_{b^l}\right)1_{\{K_1<b^l\}}\right]1_{\{K_2<b^{lu}\}} \\
&\quad + \left[\frac{K_M-K_1}{K_M-b^l}c_{b^l}1_{\{K_1>b^l\}} + c_{K_1}1_{\{K_1<b^l\}}\right]1_{\{K_2>b^{lu}\}} \\
\underline{h}^* &= -c_{K_2}\;1_{\{K_2>b^{lu}\}}
\end{aligned}$$

where b^l (b^{ll}, b^{lu}) and b^u (b^{ll}, b^{lu}) are defined as in Lemma 4.1. In particular, $\phi^* \in [-1,1]$.

Proof: The proof is analogous to the one of Propodition 4.5. □

Proposition C.2 (Tractable Robust Hedge of Bullish Call Spread with Additional Instrument (iii)). *The cheapest tractable hedge with additional instrument c_{K_M} of a claim*

[10] Recall again that a long position is dynamically hedged at the upper volatility bound and the short position at the lower volatility bound

with payoff–function $h(x) = (x-K_1)^+ - (x-K_2)^+$ *(*$K_1 < K_2 < K_M$*) is given by*

$$\begin{aligned}
\phi^* &= \left[-\frac{K_2-K_1}{K_2-K^*}\frac{b^u-K_2}{b^u-K_M}1_{\{K^*<K_1\}} - \frac{b^u-K_2}{b^u-K_M}1_{\{K^*>K_1\}}\right]1_{\left\{b^{ul}<K_2\right\}} \\
\bar{h}^* &= \left[\left(\frac{K_2-K_1}{K_2-K^*}c_{K^*} + \frac{K_2-K_1}{K_2-K^*}\frac{K_M-K_2}{b^u-K_M}c_{b^u}\right)1_{\{K^*<K_1\}}\right. \\
&\quad \left. + \left(c_{K_1} + \frac{K_M-K_2}{b^u-K_M}c_{b^u}\right)1_{\{K^*>K_1\}}\right]1_{\left\{b^{ul}<K_2\right\}} \\
&\quad + \frac{K_2-K_1}{K_2-\min\{a^*,K_1\}}c_{\min\{a^*,K_1\}}1_{\left\{b^{ul}>K_2\right\}} \\
\underline{h}^* &= \frac{-(K_2-K_1)}{K_2-\min\{a^*,K_1\}}c_{K_2}1_{\left\{b^{ul}>K_2\right\}}
\end{aligned}$$

where K^* *solves*

$$\frac{\partial u(t,x;c_{K^*})}{\partial K^*} = \frac{u(t,x;c_{K^*}) - \left[\frac{b^u-K_2}{b^u-K_M}\ M_t(c_{K_M}) + \frac{K_2-K_M}{b^u-K_M}\ u(t,x;c_{b^u})\right]}{K^*-K_2},$$

a^* *is defined as in Proposition 4.5 and* b^l *(*b^{ll}, b^{lu}*) and* b^u *(*b^{ll}, b^{lu}*) are defined as in Lemma 4.1. In particular,* $\phi^* \in [-1,1]$*.*

PROOF: The proof is analogous to the one of Propodition 4.5. □

References

Avellaneda, M. and Parás, A. (1996), Managing the volatility risk of portfolios of derivative securities: the Lagrangian uncertain volatility model, *Applied Mathematical Finance* **3**(1), 21–53.

Avellaneda, M., Levy, A. and Parás, A. (1995), Pricing and Hedging Derivative Securities in Markets with Uncertain Volatilities, *Applied Mathematical Finance* **2**(2), 73–88.

Bakshi, G., Cao, C. and Chen, Z. (1997), Empirical Performance of Alternative Option Pricing Models, *Journal of Finance* **52**(5), 2003–2049.

Bertsimas, D., Kogan, L. and Lo, A. (1998), When is Time Continuous, *Technical report*, Massachusetts Institute of Technology (MIT).

Bossy, M., Gibson, R., Lhabitant, F., Pistre, N. and Talay, D. (2001), A Methodology to Analyze Model Risk with an Application to Discount Bond Options in a Heath-Jarrow-Morton Framework, Working Paper.

Bossy, M., Gibson, R., Lhabitant, F., Pistre, N., Talay, D. and Zheng, Z. (2000), Volatility Model Risk measurement and strategies against worst case volatilities, *Journal de la Socit Franaise de Statistique* **141**(1-2), 73–86.

Boyle, P. and Emanuel, D. (1980), Discretely Adjusted Option Hedges, *Journal of Financial Economics* **8**, 259–282.

Branger, N. and Mahayni, A. (2006), Tractable Hedging– An Implementation of Robust Hedging Strategies, *Journal of Economic, Dynamics and Control* **30**, 1937–1962.

Broadie, M., Chernov, M. and Johannes, M. (2005), Model Specification and Risk Premiums: Evidence From Futures Options, Working Paper.

Carr, P. and Wu, L. (2002), Static Hedging of Standard Options, Working Paper.

Duffie, D., Pan, J. and Singleton, K. (2000), Transform Analysis and Asset Pricing for Affine Jump Diffusions, *Econometrica* **68**, 1343–1376.

El Karoui, N. and Quenez, M.-C. (1995), Dynamic Programming and Pricing of Contingent Claims in an Incomplete Market, *SIAM J. Control and Optimization* **33**(1), 29–66.

Eraker, B., Johannes, M. and Polson, N. (2003), The Impact of Jumps in Volatility and Returns, *Journal of Finance* **58**(3), 1269–1300.

Heston, S. (1993), A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies* **6**(2), 327–343.

Hull, H. and Suo, W. (2002), A Methodology for Assessing Model Risk and its Application to the Implied Volatility Function Model, *Journal of Financial and Quantitative Analysis* **37**(2), 297–318.

Hull, J. and White, A. (1987), The Pricing of Options on Assets with Stochastic

Volatilites, *Journal of Finance* **42**(2), 281–300.

Jiang, G. and Oomen, R. (2001), Hedging Derivatives Risks – A Simulation Study, Working Paper.

Lyons, T. (1995), Uncertain Volatility and the Risk-free Synthesis of Derivatives, *Applied Mathematical Finance* **2**, 117–133.

Merton, R. (1976), Option Pricing When Underlying Stock Returns are Discontinuous, *Journal of Financial Economics* **3**, 125–144.

Vargiolu, T. (2001), Existence, Uniqueness and Smoothness for the Black–Scholes–Barenblatt Equation, *Technical report*, Department of Pure and Applied Mathematics of the University of Padova.