The Impact of White Noise in PD Estimations on Banks' Capital Requirements according to Basel II[†]

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Abstract

The IRB Approach of Basel II allows banks to use internal rating models to determine their credit risk's capital requirements. These charges are concave functions of the PD. Thus, by using an estimator for the PD which is contaminated by some noise, banks arrive at lower expected capital requirements than by using a more accurate estimator. In this article, we examine the scope of and the incentives generated by this effect both theoretically and via simulations. We also discuss according regulatory directives and derive some advice for supervisors to reveal a possible utilization of this unintended opportunity.

Key Words: Banking Supervision, Basel II, IRB Approach, Capital Requirements, Estimation Errors.
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1 Introduction

Banks are required to hold regulatory capital as a cushion against the risks involved with their business. The first capital accord, published by the Basel Committee on Banking Supervision in 1988 and later named Basel I, set up some first principles, particularly for the capital which has to be set aside for loan losses.[1] Subsequently it was amended by rules for market risk[2] and eventually it is being replaced by the new capital accord, Basel II.[3]

Without going into any details, it can be said that reducing the gap between regulatory capital and economic capital[4] was one of the key objectives of the new accord.[5] Apart from external ratings, banks' internal ratings may serve as the leading indicators of debtor's credit worthiness and the resulting probabilities of default (PDs) therefore as basic determinants of the regulatory capital to be held against the loan. Regulators have promised to provide incentives for improving internal rating systems.[6]

The main purpose of this paper is to show that the way regulatory capital is calculated does not necessarily satisfy this promise. More specifically, we are able to prove that a bank may need a higher amount of regulatory capital when it reduces the white noise in its PD estimations, i.e., when it increases the accuracy. Thus, less noisy PD estimations are punished by a higher capital charge and are therefore, from this perspective, less attractive. Within certain limits, it may even be attractive for banks to add white noise to reduce their capital requirements.

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[1] Basel Committee on Banking Supervision (1988).

[2] Basel Committee on Banking Supervision (1996).

[3] Basel Committee on Banking Supervision (2006a).
While the rules formally as such are only binding for large international banks, over the years more than 100 countries have integrated Basel I into their national regulation. Something similar may happen with respect to Basel II.

[4] In a nutshell, economic capital is the amount of capital banks hold voluntarily, e.g., because they want to achieve a certain rating themselves.

[5] See Basel Committee on Banking Supervision (2006a), para. 18.

[6] See e.g. Basel Committee on Banking Supervision (2006b), p. 3, and Basel Committee on Banking Supervision (2001), p. 1.

What is the reason for this apparently odd result? Leaving aside several specifical rules and technicalities, the capital charge for a loan is a concave function of the debtor's PD. Making use of some well-known statistical results involving second order stochastic dominance which previously were applied, among others, in risk measurement, our claim will become rather obvious.[7] We will demonstrate how large the extent of potential changes in capital requirements is by means of a simulation analysis.

Our paper is organized as follows: In Section 2, we provide a brief review of the related literature. In Section 3, the possibility of reducing capital charges in the Internal Ratings Based Approach (IRBA) of Basel II is shown by applying second order stochastic dominance. We further reveal conceivable incentives for banks: Firstly, they could *abstain from improving* an existing (noisy) rating system, and secondly, they could *actively worsen* their PD estimators by adding noise. We sacrifice realism, possibly, to facilitate our presentation by assuming that noise is added on purpose. This enables us to choose a simple functional form for the noise and also allows the derivation of suitable constraints. Possibilities of implementing such a modification are shown. In Section 4, the extent of the reduction effect is quantified in several simulations. These include the analysis of single loans, homogeneous portfolios, and heterogeneous portfolios. In the latter case, modifications of *all* PDs and solely of the PDs of the *1 % biggest loans* are both examined. Although derived the other way round, simulations can also serve as examples for the additional capital charges that result from an improvement of a noisy rating system. In Section 5, supervisory requirements concerning internal rating models in the IRBA are examined. We check whether adding noise is compatible with these guidelines. Obviously it is not and therefore we show

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[7] By the same token, a convex function would remove the incentive to differentiate PDs altogether and hence could not be used by the Basel Committee. We owe this insight to Guenter Franke.

possibilities for banks to hide such modifications, but also give some advice to supervisors on revealing them. Section 6 briefly concludes.

2 Related literature

A discussion of the concavity of the capital requirements and resulting effects can be found in several papers. However, most authors focus on issues different from ours:

Broeker (2004) examines the impact of the discriminative power of a rating system on the capital requirements. Mainly by examples, he shows that a higher discriminative power leads to reduced capital charges. Rauhmeier and Scheule (2005) examine the same effect from a mathematical perspective and reach the same conclusion.

Jankowitsch *et al.* (2003) examine the impact of rating granularity, i.e. the number of rating classes, on banks' capital requirements. They find that capital charges decline with the fineness of the rating system and also show that, for a given number of rating classes, there exists an optimal choice of the classes' boundaries in the sense of minimizing the capital charges.

Kiefer and Larson (2004) are dealing with a similar topic: They examine the trade-off between bucket size and estimation accuracy of the average PD of a bucket. The larger the size of a bucket is, the more accurate are the estimations of the average PD, but the less relevant is this PD because of the increased heterogeneity of the loans in the bucket. This trade-off is examined in the context of economic capital allocation according to Basel II (where the concavity of the risk weights function is considered, as well).

In a further paper Jankowitsch *et al.* (2007) examine the economic value of credit rating systems. They model the quality of a rating system by several parameters, like forecasting accuracy of PD estimations and rating class structure, and measure the effects of adverse selection in a one-period setup by examining customer elasticity.[8] This is quantified by simulations. However, the capital reduction resulting from less accurate PD forecasts is neither examined separately nor proved mathematically.

Our analysis, therefore, supplements the work of Jankowitsch *et al.* (2007) by focussing solely on the impact of an, accidental or deliberate, inaccuracy of PD estimations. Indeed, this is no contradiction to the effect described by Broeker (2004) and Rauhmeier and Scheule (2005). Their reasoning implies the application of more rating classes when increasing the discriminative power of a rating system. In contrast, we consider a fixed rating granularity.

3 Theoretical analysis

3.1 The general reduction effect

Consider a bank which has granted n loans. Then, it has to hold capital against the credit risk. According to the IRBA of Basel II, the capital requirement factor[9] K_j for the unexpected loss of loan j is defined by

$$\begin{aligned} K_j = LGD_j \quad & \cdot \quad \left[N\left(\frac{N^{-1}(PD_j) + \sqrt{\rho_j} \cdot N^{-1}(0.999)}{\sqrt{1-\rho_j}} \right) - PD_j \right] \\ & \cdot \quad \frac{1}{1 - 1.5 \cdot b(PD_j)} \cdot \underbrace{[1 + (M - 2.5) \cdot b(PD_j)]}_{\neq 1 \text{ only for advanced IRBA}} \end{aligned}$$

with the cumulative density function N of the standard normal distribution and the following denotations with respect to loan j:

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[8] The customer elasticity quantifies the effect of customers possibly leaving the bank when being classified into a wrong rating class.

[9] To enhance clarity, in the context of this paper the term "capital requirement factor" is used for the expression "capital requirement" in Basel II.

$$\begin{aligned} LGD_j &= \text{loss given default}, \\ PD_j &= \text{one-year probability of default}, \\ \rho_j &= \text{correlation}, \\ b(PD_j) &= \text{maturity adjustment}, \\ M_j &= \text{maturity}. \end{aligned}$$

More precisely, Basel II specifies formulas for the correlation, maturity adjustment, and maturity, which depend on PD_j, as well. In case of corporate exposures, the correlation also depends on the annual turnover S_j.[10]
As a function of PD_j (with fixed values for all other factors), the capital requirement factor $K_j = K_j(PD_j)$ is concave, as can be seen in Fig. 1.[11]

This holds for the overall capital requirement CR_j for asset j, too:

$$CR_j(PD_j) = 8\,\% \cdot 12.5 \cdot EAD_j \cdot K_j(PD_j) = EAD_j \cdot K_j(PD_j),$$

where EAD_j denotes the exposure at default of loan j.[12]
Henceforth, we consider all factors as fixed, except the probabilities of default. Thus, the capital requirement for loan j can be written as $CR_j(PD_j)$ and, in sum, the bank's capital charge for credit risk amounts to[13]

$$CR = \sum_{j=1}^{n} CR_j(PD_j) = \sum_{j=1}^{n} EAD_j \cdot K_j(PD_j).$$

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[10] The according formulas for correlation, maturity adjustment, and maturity are listed in Appendix A.1.

[11] The used parameters in Fig. 1 are: LGD = 45 % and M = 2.5; the size S is specified in € m.

[12] According to para. 43 of Basel II, the difference between the total expected loss $\sum_j EL_j = \sum_j EAD_j \cdot LGD_j \cdot PD_j$ and the value adjustments diminishes the available capital, which has to be considered in the examination, basically. Since the addition of a white noise random variable to the PD does not change the mean of the total expected loss (c.p.), this effect is neglected in the context of this paper.

[13] In fact, the capital charge is determined by the *average* PD of the rating class the borrower belongs to. In this paper this fact is neglected because precise information about the rating class boundaries would be necessary, otherwise. Whereas, the primary concern of our paper is to analyze the reduction effect in a general framework.

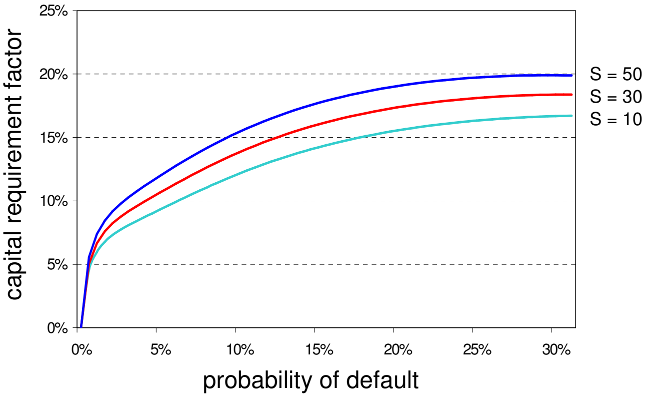

Figure 1: The capital requirement factor K_j for corporate exposures as a function of the PD.

The real probabilities of default $PD_1, ..., PD_n$ are unknown, in fact, and can only be estimated. Therefore, the bank uses an (ideally unbiased) estimator $R(A_1^j, ..., A_k^j) =: R_j$, $j = 1, ..., n$, for PD_j which is [0,1]-valued and estimates PD_j by m random factors $A_1^j, ..., A_m^j$, e.g. balance sheet ratios of the borrower.[14] The bank's overall capital charge is then determined by $\sum_{j=1}^{n} CR_j(r_j)$, where $CR_j(r_j)$ denotes a realization of $CR_j(R_j)$.
The *expected* capital requirements amount to

$$E(\sum_{j=1}^{n} CR_j(R_j)) = \sum_{j=1}^{n} E(CR_j(R_j)).$$

In the following we show that the expected charges are lower when using an *inaccurate* estimator for the PD than when using a more *accurate* one with the same mean, c.p. This holds both for single loans and the entire loan portfolio.

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[14] Since the factor distributions are borrower-specific, they carry the subscript j.

For this purpose the expressions “accurate” and “inaccurate”, respectively, need to be specified. Let R_j denote an estimator for PD_j and X_j another one which results from R_j in the following way:[15]

$$X_j \sim R_j + Z_j,$$

where Z_j is a random variable with[16]

$$E(Z_j|R_j) = 0 \;\; a.s.. \tag{1}$$

Then we say “R_j is more accurate than X_j” and “X_j is more inaccurate than R_j”, respectively. This interpretation is underlined by the fact that Z_j is a white noise random variable with respect to R_j:

Lemma 1.

For $j \in \{1, ..., n\}$ let X_j and R_j be two random variables. Let X_j result from R_j by $X_j \sim R_j + Z_j$ where Z_j is a random variable with the property $E(Z_j|R_j) = 0$ $a.s.$. Then,

i) $E(Z_j) = 0$,

ii) R_j and Z_j are uncorrelated.

Hence, Z_j is a white noise random variable with respect to R_j.

Proof.

Let $j \in \{1, ..., n\}$. The properties of conditional expectations[17] directly yield statement i): $E(Z_j) = E(E(Z_j|R_j)) = 0$.

The lack of correlation can bee seen as following:

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[15] The notation “$X \sim Y$” for two random variables X and Y means that X and Y are identically distributed.

[16] “a.s.” is an abbreviation for “almost surely” and means that the equation holds with probability 1.

[17] See e.g. Billingsley (1995).

$$\begin{aligned}
Cov(Z_j, R_j) &= E(Z_j \cdot R_j) - E(Z_j) \cdot E(R_j) \\
&\overset{E(Z_j)=0}{=} E(Z_j \cdot R_j) \\
&= E(E(Z_j \cdot R_j | R_j)) \\
&= E(R_j \cdot E(Z_j | R_j)) \\
&\overset{E(Z_j|R_j)=0 \;\; a.s.}{=} 0. \qquad \square
\end{aligned}$$

Hence, property (1), i.e. $E(Z_j|R_j) = 0 \;\; a.s.$, implies that Z_j is a white noise random variable with respect to R_j. Nevertheless, for our purpose, it is essential to define the relation "R_j is more accurate than X_j" exactly as we did since the reverse conclusion does not hold[18] and property (1) is a requirement for the following theorem – the major result of our paper: The use of a more inaccurate PD estimator X_j instead of a more accurate one R_j leads to a reduction in the bank's expected capital requirements. This holds both for single loans and the entire loan portfolio.

Theorem 1.
For $j \in \{1, ..., n\}$ let X_j and R_j be estimators for PD_j. Furthermore, let X_j fulfill $X_j \sim R_j + Z_j$, where Z_j is a random variable with $E(Z|R_j) = 0 \;\; a.s.$. Then,

i) $E(K_j(X_j)) \leq E(K_j(R_j))$, and consequently,
$E(CR_j(X_j)) \leq E(CR_j(R_j))$,

ii) $E(\sum_{j=1}^{n} K_j(X_j)) \leq E(\sum_{j=1}^{n} K_j(R_j))$, and hence,
$E(\sum_{j=1}^{n} CR_j(X_j)) \leq E(\sum_{j=1}^{n} CR_j(R_j))$.

Proof.
Let $j \in \{1, ..., n\}$. Without restriction we assume $P(R_j \in [0, 1]) = P(X_j \in$

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[18] as can bee shown with simple examples, see e.g. Rothschild and Stiglitz (1970).

$[0,1]) = 1$ since both random variables are estimators for $PD_j \in [0,1]$. Then, $K_j(R_j)$ and $K_j(X_j)$ are *a.s.* defined.[19] According to Rothschild and Stiglitz (1970) we obtain the relation "X_j is equal to R_j plus noise" due to the condition $E(Z_j|R_j) = 0$ *a.s.* and the definition of X_j. Finally, the concavity of K_j and the property $CR_j = EAD_j \cdot K_j$ yield the proposition for a single loan j, as is shown by Rothschild and Stiglitz (1970), as well.[20]
The validity of the proposition for the entire portfolio is obvious.

□

In this paper we focus on white noise as error terms in PD estimations since their mean preserving property is less likely to be detected by the backtesting procedures envisaged by Basel II.[21] When using general error terms this property could be violated and the modified model could therefore be more easily denied by the supervisors.

3.2 Incentives for banks

In the following, we assume that banks have a limited amount of available equity or least have an interest to lower their capital requirements. The effect described in the previous section leads to reduced expected capital requirements when using an – intentionally or not – inaccurate PD estimator instead of a more accurate one. This involves, for supervisory purposes, undesirable incentives and thwarts the main idea of the IRB approach: Banks might not only have an interest in *not improving* existing rating models. They might even try to *intentionally worsen* estimators to reduce their capital charges. The incentive not to improve an existing rating model is in our opinion very important and alarming, in particular from the supervisors' point of view.

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[19] where we set $K_j(0) = K_j(1) = 0$, see also the proof of Corollary 1.

[20] The property used in the proof often is referred to as *second order stochastic dominance* and *second degree stochastic dominance*, respectively, see e.g. Levy (1998).

[21] See also Section 5.2.

We will bear this aspect in mind (in Section 4, we will give several simulation results that show the additional capital charges resulting from the improvement of an existing rating model), but the focus of our examination will be lying on the incentive to worsen existing estimators. This is on the one hand due to the possibility of choosing the noise at our discretion, and on the other hand due to the fact that banks have to consider an important constraint when constructing a suitable white noise random variable: According to Theorem 1 a deterioration is possible by adding a random variable Z_j with $E(Z_j|R_j) = 0 \;\; a.s.$ to a given PD estimator R_j. The property $X_j := R_j + Z_j \in [0,1] \;\; a.s.$ is a constraint since the PD which is to be estimated is [0,1]-valued.[22] Therefore, Z_j cannot be chosen independently from the realizations r_j of R_j.[23]

In the next section we examine how to cope with this task.

3.3 Making the effect work

The first result in this section poses a sufficient condition for a noise leading to admissible PD modifications (in the sense of a.s. lying in the interval [0,1]).

Lemma 2.

For $j \in \{1, ..., n\}$ let r_j be a realization of R_j and Z_j be a random variable with the property

$$P^{Z_j|R_j=r_j}([-r_j, 1-r_j]) \;=\; 1.$$

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[22] Note that for corporate exposures, a minimum PD of 0.03 % is to be used. In the context of this paper, this fact is neglected since we examine the reduction effect in a general framework and the procedure and results are analogous when regarding 0.03% as lower boundary.

[23] For example, if $r_j = 0$, there is (a.s.) only one possibility for the choice of Z_j: A random variable Z_j with $Z_j \equiv 0 \; a.s.$ is the only white noise random variable with $E(Z_j|R_j) = 0 \; a.s.$ and $r_j + Z_j \in [0,1] \; a.s.$. Whereas, if $r_j \in (0,1)$, the constraint of the modified PD lying in the interval [0,1] is achievable by suitable white noise random variables with positive variance, too.

Then, $X_j := R_j + Z_j \in [0,1] \;\; a.s.$. This holds in particular for Z_j with

$$P^{Z_j|R_j=r_j}([-r_j^O, r_j^O]) \;=\; 1,$$

where $r_j^O := min\{r_j; 1-r_j\}$.[24]

Proof.
Let r_j be a realization of R_j and z_j a realization of Z_j. Then we obtain $r_j+z_j \in [r_j-r_j, r_j+1-r_j] = [0,1]$. As $[-r_j^O, r_j^O] \subset [-r_j, 1-r_j]$, the second statement holds. $\square$

That means if a bank intends to worsen its PD estimator in order to reduce the expected capital requirements it should add a random variable Z_j with the properties

i) $E(Z_j|R_j) = 0 \;\; a.s.$ and

ii) $P^{Z_j|R_j=r_j}([-r_j, 1-r_j]) \;=\; 1$

to an existing PD estimator R_j. Property i) then provides the satisfaction of the requirements of Theorem 1 and property ii) makes sure that $X_j := R_j + Z_j$ really can pass for a PD estimator in the sense of (a.s.) producing values in [0,1]. An example for such a random variable is Z_j with $P^{Z_j|R_j=r_j}([-r_j^O, r_j^O]) \;=\; 1$, as seen in Lemma 2.
Normal distributions, e.g., do not meet criterion ii) and are therefore unsuitable for Z_j.

At first glance it is astonishing that, although depending on R_j, Z_j is a white noise random variable with respect to R_j, as is shown in Lemma 1. That

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[24] The superscript "O" indicates that the overall range of the interval [0,1] is used for the construction of r_j^O.

means in particular, Z_j and R_j are uncorrelated. However, this is no contradiction to the dependence of Z_j and R_j since the correlation only concerns the *linear* relationship of two random variables.

We now give an example for such a random variable Z_j: Consider Z_j^O with the factorized conditional distribution

$$P^{Z_j^O|R_j=r_j} = U(-r_j^O, r_j^O), \tag{2}$$

where $U(a,b)$ denotes the uniform distribution on the interval $[a,b]$.
Then, property i) is fulfilled since

$$E(Z_j^O|R_j = r_j) = \int_{[-r_j^O, r_j^O]} x dP^{Z_j^O|R_j=r_j}(x) = 0$$

for all r_j, and thus, $E(Z_j^O|R_j) = E(Z_j^O|R_j = \cdot) \circ R_j = 0 \;\; a.s.$. Property ii) is obvious.

As can be seen, adding this particular Z_j^O to a given PD estimation r_j creates values which are uniformly distributed on the interval $[0, 2r_j]$ and $[1-2r_j, 1]$, respectively.

The choice of a uniform distribution in the foregoing example is arbitrary, indeed. Though, it proves to be a compromise between two extreme cases:
On the one hand, there is no reduction effect in capital requirements when adding a deterministic white noise random variable Z_j with $P(Z_j = 0) = 1$.
On the other hand, the reduction effect is maximized when using the following random variable Z_j^{max}:

Proposition 1.

For $j \in \{1, ..., n\}$ let r_j be a realization of R_j and Z_j^{max} a random variable with the factorized conditional distribution

$$P^{Z_j^{max}|R_j=r_j}(\{-r_j\}) = 1 - r_j = 1 - P^{Z_j^{max}|R_j=r_j}(\{1 - r_j\}). \tag{3}$$

Then the following two effects are observed when using $X_j^{max} = R_j + Z_j^{max}$ as a modified PD estimator:

i) The expected reduction of capital charges for loan j under the constraints $E(Z_j^{max}|R_j) = 0$ *a.s.* and $X_j^{max} \in [0, 1]$ *a.s.* is maximized.

ii) Loan j is classified either in the best or in the worst rating class. Concerning the entire loan portfolio, the expected number of credits in the best rating class amounts to $\sum_{j=1}^{n}(1 - E(R_j))$ and in the worst class $\sum_{j=1}^{n} E(R_j)$, respectively.

Proof.

Let $j \in \{1, ..., n\}$.

Statement i):

The properties $E(Z_j^{max}|R_j) = 0$ *a.s.* and $X_j^{max} = R_j + Z_j^{max} \in [0, 1]$ *a.s.* are obvious. Now, let Y_j be a random variable which fulfills these conditions, too. Then we obtain "X_j^{max} has more weight in the tails than $X_j := R_j + Y_j$" according to the definition of Rothschild and Stiglitz:[25] The distribution of X_j^{max} is obtained by a mean preserving spread which shifts the entire probability mass to the boundary points {0,1}.[26] Because of $R_j \in [0, 1]$ *a.s.* the probability mass is shifted *outwards*. Since a further shift of probability mass outwards under the constraints $X_j^{max} \in [0, 1]$ *a.s.* and $X_j \in [0, 1]$ *a.s.*,

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[25] See Rothschild and Stiglitz (1970).

[26] This technique, a mean preserving shift of all probability mass into the interval boundaries, is used, among others, in the potential theory (see e.g. Doob (1984)) and is called *balayage*.

respectively, is not possible the random variable Z_j^{max} with the in (3) formulated factorized conditional distribution is the only one with the properties $X_j^{max} \in \{0, 1\}$ $a.s.$ and $E(Z_j^{max}|R_j) = 0$ $a.s.$ (and $E(X_j^{max}) = E(R_j)$, respectively). E.g., a further mean preserving assignment of probability mass outwards (i.e. away from the mean) but within the interval [0,1] is not possible. Due to the relation formulated above and according to Rothschild and Stiglitz (1970) we obtain

$$E(CR_j(X_j^{max})) \leq E(CR_j(X_j)),$$

which yields the proposition.

Statement ii):

The property $X_j^{max} \in \{0, 1\}$ $a.s.$ is obvious. Let I_0 denote the interval corresponding to the best rating class, and I_1 that one corresponding to the worst rating class.[27] Then, the expected number of loans in the best class amounts to[28]

$$\begin{aligned} E(\sum_{j=1}^{n} \; {}_{I_0}(X_j^{max})) &= \sum_{j=1}^{n} E(\; {}_{I_0}(X_j^{max})) = \sum_{j=1}^{n} P(X_j^{max} \in I_0) \\ &= \sum_{j=1}^{n} P(X_j^{max} = 0) = \sum_{j=1}^{n} P(R_j + Z_j^{max} = 0) \\ &= \sum_{j=1}^{n} \int_{[0,1]} P(Z_j^{max} = -r_j | R_j = r_j) dP^{R_j}(r_j) \end{aligned}$$

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[27] For this purpose we assume the existence of at least two rating classes so that the intervals I_0 and I_1 are disjunct with the properties $0 \in I_0$ and $1 \in I_1$.
According to the requirements concerning the IRBA in Basel II this is no restriction because the existence of at least eight rating classes is demanded.

[28] For a set A, $\;{}_A$ denotes the indicator function:

$$\;{}_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \int_{[0,1]} (1 - r_j) dP^{R_j}(r_j) = \sum_{j=1}^{n} (1 - \int_{[0,1]} r_j dP^{R_j}(r_j)) \\
&= \sum_{j=1}^{n} (1 - E(R_j)),
\end{aligned}
$$

which is obtained by making use of Fubini's Theorem. Thus, the expected number of loans in the worst rating class amounts to

$$n - \sum_{j=1}^{n} (1 - E(R_j)) = \sum_{j=1}^{n} E(R_j).$$

□

When adding differently distributed white noise random variables, the expected extent of the reduction differs. By considering the random variable defined in (3), we are able to quantify the *maximum* extent of the expected reduction, i.e. there exists no other white noise random variable which leads to a more distinctive expected reduction.

Corollary 1.
The minimum expected capital requirement in a modified model which results by adding white noise random variables is 0.

Proof.
Let $j \in \{1, ..., n\}$ and r_j be a realization of R_j. As seen in Proposition 1, the reduction effect maximizes under the constraints $E(Z_j|R_j) = 0$ $a.s.$ and $X_j \in [0, 1]$ $a.s.$ when adding $Z_j = Z_j^{max}$ with the factorized conditional distribution in (3). For the resulting $X_j = r_j + Z_j$, which is {0,1}-valued, we can compute the expected capital charge: Since, by construction of Z_j, $P(X_j = 0) = 1 - r_j$ and $P(X_j = 1) = r_j$, we obtain

$$E(CR_j(X_j)) = (1 - r_j) \cdot CR_j(0) + r_j \cdot CR_j(1).$$

The values $CR_j(0) = EAD_j \cdot K_j(0)$ and $CR_j(1) = EAD_j \cdot K_j(1)$ cannot be computed since neither $N^{-1}(0)$ nor $N^{-1}(1)$ exists. But they can be determined by extensions in a limit consideration: As $\rho(x) \to const \in (0,1)$ for $x \to 0$ and $x \to 1$ for all asset classes, respectively, and $\frac{1}{1-1.5\cdot b(x)} \cdot [1 + (M - 2.5) \cdot b(x)] \to const \in$ for $x \downarrow 0$ and $x \to 1$, respectively,[29] we obtain

$$\begin{aligned} \lim_{x \downarrow 0} K_j(x) &= 0 \quad \text{and} \\ \lim_{x \uparrow 1} K_j(x) &= 0 \end{aligned}$$

(see also Fig. 2). With $K_j(0) = 0$ and $K_j(1) = 0$ as continuous extensions of K_j, we obtain $E(CR_j(X_j)) = EAD_j \cdot E(K_j(X_j)) = 0$. $\square$

It seems quite remarkable to us that a reduction of the risk weight to 0 is – theoretically – possible. The reason is the shape of the capital requirement factor near 1: K_j is *not* monotonically increasing (as could be guessed) but declines to 0 when reaching the upper interval boundary 1. This is due to the purpose of determining the capital requirements for the *unexpected* loss (which declines for borrowers with high default probabilities) and can be seen in Fig. 2.[30]

When using other distributions for $P^{Z_j|R_j=r_j}$, like the uniform distribution in (2), the shift effect of Proposition 1 ii) towards the outer rating classes occurs, too, but it is less distinct. A numerical example for the portfolio's structural change which occurs when adding Z_j as defined in (2) is provided in the next section.

A heuristic explanation of the structural change in the credit portfolio *despite of* the mean preserving property of the added white noise random variables is explained in the following. Consider for example uniformly distributed white

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[29] This follows from the rule of de l'Hospital.

[30] Fig. 2 shows the capital requirement factor for corporate exposures. The used parameters are: LGD = 45 %, S = 50 (€ m), M= 2.5.

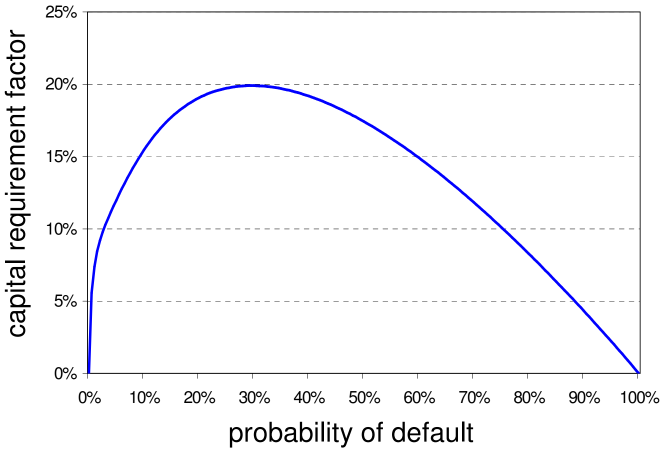

Figure 2: The capital requirement factor K_j as a function of the PD.

noise random variables. Then, original PDs that are located near the boundary 0 (and 1, respectively) remain after the addition of white noise in the area at this boundary.

Whereas, the default probability PD $= 0.5$, e.g., can reach every area in the interval $[0, 1]$ with the same probability when being modified. In particular, it can reach the area at the boundaries, as well. Thereby, the number of observed modified PD in the outer classes may increase. A further reason for the structural change is the exponential assignment of PDs to the according rating classes (which yields a shift mainly towards *better* classes).

As a structural change of the credit portfolio might be undesirable on the part of the banks, they might search for possibilities to prevent this change. A simple possibility is to apply suitable white noise random variables that create values X_j a.s. lying in the *same* rating class as the original PD.[31] This can be achieved by considering the boundaries of the according class for the construction of the white noise random variable. Analogously to Lemma 2, we obtain:

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[31] Note that this proceeding is only useful if the bank reports the PDs (and not the corresponding rating classes) to the supervisors.

Lemma 3.

Let the realization r_j of R_j be associated with rating class k, i.e. $r_j \in I_k = [a_k, b_k]$, then a random variable Z_j with

$$P^{Z_j|R_j=r_j}([-r_j^C, r_j^C]) \;=\; 1,$$

where $r_j^C := min\{r_j - a_k; b_k - r_j\}$,[32] is a white noise random variable with respect to R_j and yields $X_j := R_j + Z_j \in [a_k, b_k] = I_k \;\; a.s.$.

In the case of a continuous distribution $P^{Z_j|R_j=r_j}$, it is irrelevant whether the boundaries a_k or b_k belong to the interval I_k or not. This is because $P^{Z_j|R_j=r_j}(\{a_k\}) = P^{Z_j|R_j=r_j}(\{b_k\}) = 0$ and therefore the probability of reaching the next rating class equals 0. In the case of a discrete distribution, the effect of possibly reaching the next class can be prevented by a slightly different definition of Z_j. It can be achieved by defining the range

$$r_j^{C\,\prime} := min\{\frac{r_j - a_k}{2}; \frac{b_k - r_j}{2}\},$$

for example. An illustration is given in Fig. 3.

In each rating class, the maximum expected reduction for a loan j, when adding a suitable white noise random variables under the constraint of maintaining the rating class, can be determined analogously to Corollary 1:

Corollary 2.

Let the realization r_j of R_j be associated with rating class k, i.e. $r_j \in I_k = [a_k, b_k]$, then the maximum extent of the expected reduction of the capital

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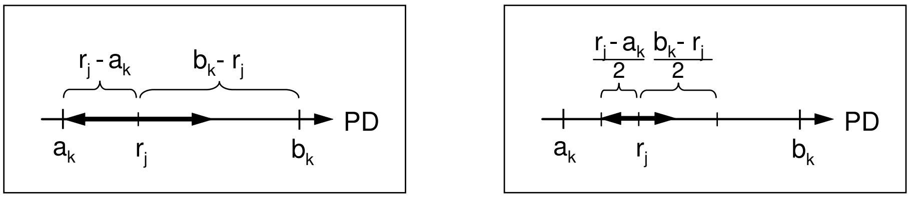
[32] The superscript "C" indicates that the class interval is used for the construction of the range r_j^C.



(a) (b)

Figure 3: Modifying the PD within the rating class. The marked arrow shows the interval where the modified PD is (a.s.) situated. In (a), the entire range $r_j^C = min\{r_j - a_k; b_k - r_j\}$ is used, in (b), the range $r_j^{C\,\prime} = min\{\frac{r_j - a_k}{2}; \frac{b_k - r_j}{2}\}$.

requirement by adding a white noise random variable Z_j under the constraint $r_j + Z_j \in I_k$ is

$$\begin{aligned} & CR_j(r_j) - [p_j \cdot CR_j(a_k) + (1 - p_j) \cdot CR_j(b_k)] \\ = \; & EAD_j \cdot \Big(K_j(r_j) - [p_j \cdot K_j(a_k) + (1 - p_j) \cdot K_j(b_k)] \Big) \end{aligned}$$

with $p_j := \frac{b_k - r_j}{b_k - a_k}$.

Proof.

With the aforementioned p_j (which in the case $a_k = 0, b_k = 1$ equals the probability $1 - r_j$ from (3)), the random variable Z_j with

$$P^{Z_j | R_j = r_j}(\{a_k - r_j\}) = p_j = 1 - P^{Z_j | R_j = r_j}(\{b_k - r_j\})$$

has analogously to Proposition 1 the following properties:

The expected capital charges for loan j minimize under the constraints $E(Z_j | R_j) = 0 \; a.s.$ and $X_j = R_j + Z_j \in [a_k, b_k] \; a.s.$ when using X_j as modified PD estimation: The property $X_j \in [a_k, b_k] \; a.s.$ is obvious and $E(Z_j | R_j) = 0 \; a.s.$ follows from the definition of p_j:

$$\begin{aligned}
E(Z_j|R_j = r_j) &= (a_k - r_j) \cdot P^{Z_j|R_j=r_j}(\{a_k - r_j\}) \\
&\quad + (b_k - r_j) \cdot P^{Z_j|R_j=r_j}(\{b_k - r_j\}) \\
&= (a_k - r_j) \cdot \frac{b_k - r_j}{b_k - a_k} + (b_k - r_j) \cdot (1 - \frac{b_k - r_j}{b_k - a_k}) \\
&= (a_k - r_j) \cdot \frac{b_k - r_j}{b_k - a_k} + (b_k - r_j) \cdot \frac{b_k - a_k - b_k + r_j}{b_k - a_k} \\
&= \frac{(a_k - r_j) \cdot (b_k - r_j)}{b_k - a_k} - \frac{(b_k - r_j) \cdot (a_k - r_j)}{b_k - a_k} \\
&= 0 \text{ for all } r_j,
\end{aligned}$$

and hence, $E(Z_j|R_j) = 0$ *a.s.*. Finally, the property of minimizing the expected capital charges follows analogously to the proof of Proposition 1 by making use of the balayage technique. □

Note that the maximum expected reduction is an expected value, yet. It is therefore possible to observe reductions that are even more distinctive.
In the context of the simulations in the next section numerical results for the potential reduction in each class are provided, too.

4 Simulation analysis

4.1 Outline

As Theorem 1 does not quantify the possible reduction of the bank's expected capital requirements we present results of several simulations. For this purpose, we consider the two white noise random variables that have been presented in the preceding section: Z_j^O and Z_j^C.
Note that there are two different possibilities of interpreting the addition of a random variable Z_j with the properties $E(Z_j|R_j) = 0$ *a.s.* and $P^{Z_j|R_j=r_j}([-r_j, 1 - r_j]) = 1$. Firstly, when assuming that R_j is the original estimator and $R_j + Z_j$ the worsened one, we quantify the reduction that re-

sults from a *deterioration* of the PD estimations. Secondly, when interpreting the noisy estimator $R_j + Z_j$ as the original one and the "pure" estimator R_j as its *improvement*, we quantify the additional capital charges that result from the improvement of an existing model. The addition of the random variable Z_j^O can be interpreted in both ways, indeed.[33]

To examine the reduction effect we consider corporate exposures and abstract from the impact of the parameters EAD, LGD, M, and S by setting $EAD_j = 1$, $LGD_j = 45\%$, $M_j = 2.5$, and $S_j = 50$ (€ m) for all j. Thereby, we can separate the reduction effect from other influences.

In a first setup, we examine the reduction effect for single loans. Then, in a second setup, we consider the reduction effect in a homogeneous portfolio context. Finally, in a third setup, we take a look at heterogeneous portfolios.

We further consider the ratings classification which is shown in Table 1 and can be considered representative for a classification of PDs.[34] Note that our examination is based on the neglect of rating classes as we consider banks reporting *probabilities of default.* Nevertheless, the PDs' classification to rating classes before and after a modification is worthwhile to look at because modifications of the PDs may lead to changes in credit portfolios' structures (i.e. changes of the number of loans in each class).

4.2 Effects for single loans

Simulation design.

To examine the reduction effect for single loans, we consider a variety of *given*

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[33] Z_j^C is not suitable for this purpose as there is no reason why nature should take care of rating class boundaries. Further, the utilization of a (conditionally) normally distributedwhite noise term, which at first glance seems more intuitive, is neither suitable since normal distributions create with positive probability realizations that are negative and greater than 1, respectively, and therefore cannot represent PD estimations.

[34] See Duefinance (2007).

class	name	PD interval	
		lower boundary (incl.)	upper boundary (excl.)
1	AAA	0	0.03 %
2	AA	0.03 %	0.06 %
3	A	0.06 %	0.12 %
4	BBB	0.12 %	0.41 %
5	BB	0.41 %	1.34 %
6	B	1.34 %	7.71 %
7	CCC	7.71 %	17 %
8	D	17 %	100 % (incl.)

Table 1: Rating classification which is used for the simulations.

PDs and add 250,000 realizations of white noise random variables, each. For the given PDs, we use the PDs that are situated in the middle of the classes. Thereby, we get an impression of the reduction's extent in the considered eight classes.

For each of the given PDs, we first determine the capital requirement factor and the corresponding rating class. Then, we add 250,000 realizations z_j^O of the white noise random variable Z_j^O with the factorized conditional distribution in (2), i.e. the uniform distribution with maximum range (under the constraints $E(Z_j^O|R_j) = 0$ $a.s.$ and $X_j^O = R_j + Z_j^O \in [0,1]$ $a.s.$[35]). Again, we calculate the requirement factor and rating class, but in this case for each PD modification. To estimate the *expected* reduction, we compute the mean of these factors, the according reduction, and additionally the average rating class of the modified PDs.

The same procedure is done for the white noise random variable Z_j^C with the factorized conditional distribution

$$P^{Z_j^C|R_j=r_j} = U(-r_j^C, r_j^C),$$

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[35] Note that in our simulations, the PD estimation R_j is considered deterministic and identified with PD_j.

i.e. a uniform distribution constructed by taking into account the rating class boundaries.[36]

Results.

With the denotations C_j for the rating class of PD_j, $\varnothing K_j^O$ for the average capital requirement factor of the PDs modified by Z_j^O, ΔK_j^O for the according average change (i.e. $\varnothing K_j^O - K_j(PD_j)$, it does not show the relative but the absolute change), and $\varnothing C_j^O$ for the average rating class of the modified PDs, we obtain the following results:

PD_j	$K_j(PD_j)$	$\varnothing K_j^O$	ΔK_j^O	C_j	$\varnothing C_j^O$
0.015 %	0.763 %	0.730 %	-0.034 %	1	1
0.045 %	1.476 %	1.394 %	-0.082 %	2	2
0.090 %	2.230 %	2.100 %	-0.130 %	3	2.83
0.265 %	4.081 %	3.799 %	-0.281 %	4	3.84
0.875 %	7.031 %	6.434 %	-0.598 %	5	4.88
4.525 %	11.604 %	11.088 %	-0.516 %	6	5.93
12.355 %	16.657 %	15.422 %	-1.235 %	7	6.91
58.500 %	15.338 %	13.448 %	-1.889 %	8	8

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[36] The construction of the realizations of Z_j^O and Z_j^C is not done independently: For each realization, we draw a realization of a random variable with the distribution $U(0,1)$. Then we transform it to the desired realizations z_j^O of Z_j^O and z_j^C of Z_j^C, respectively. That means that the realizations of both white noise random variables are perfectly correlated.
The idea behind this proceeding is the following: If we drew the realizations for the construction of z_j^O and z_j^C independently, the comparison of the results (though considering the same fixed credit portfolio) would be much more difficult: Imagine, the realizations (which are always accidental) for the construction of the z_j^O were good-natured in the sense of leading to a high reduction effect. Imagine further, the ones for the construction of the z_j^C were bad-natured, by chance. Then, the comparison of the two resulting extents of the reduction would hardly be comparable. Therefore, we use the same basis for the transformation to the realizations of the white noise random variables.
In the examination of single loans this proceeding is not that important, but in the portfolio context it is, in contrast.

By the addition of realizations of Z_j^C we obtain (with similar denotations using the superscript "C" instead of "O" and $\Delta K_j^{C,max}$ for the maximum expected reduction in the corresponding class by using Z_j^C):

PD_j	$K_j(PD_j)$	$\varnothing K_j^C$	ΔK_j^C	$\Delta K_j^{C,max}$	C_j	$\varnothing C_j^C$
0.015 %	0.763 %	0.730 %	-0.034 %	0.186 %	1	1
0.045 %	1.476 %	1.469 %	-0.007 %	0.021 %	2	2
0.090 %	2.230 %	2.218 %	-0.012 %	0.035 %	3	3
0.265 %	4.081 %	4.011 %	-0.070 %	0.224 %	4	4
0.875 %	7.031 %	6.897 %	-0.134 %	0.415 %	5	5
4.525 %	11.604 %	11.484 %	-0.120 %	0.522 %	6	6
12.355 %	16.657 %	16.489 %	-0.168 %	0.477 %	7	7
58.500 %	15.338 %	13.448 %	-1.889 %	6.162 %	8	8

Interpretation.

The extent of the reduction effect is quite distinct – but it depends on the particular PD and the choice of the white noise random variable. While the average reduction by adding realizations of Z_j^O reaches from 3 bp ($PD_j =$ 0.015 %) up to 189 bp ($PD_j = 58.5$ %), the average reduction by using Z_j^C reaches from about 1 bp ($PD_j = 0.045$ %) up to 189 bp ($PD_j = 58.5$ %). Considering real extents of the EADs, this is a vast diminution of the expected capital charges (numerical examples in a portfolio context are given in the following two sections).

There are several other effects observable, which we consider worthwhile to mention:

1. The larger the PD, the more distinct the average reduction when using Z_j^O. This is intuitive due to the construction of the white noise random variable. It holds except for the PD 0.875 (and 4.525, respectively).

2. The reduction effect of the addition of Z_j^C is much less than the one concerning Z_j^O. However, in the best and worst class, they equal each other. This is a consequence of the construction of the random variables and the choice of the PDs in the middle of each class interval.

3. When adding realizations of Z_j^O, the modified PDs tend to shift towards better classes. The reason is the exponential assignment of the PDs to the classes. Exceptions are the best class (which is clear due to the absence of a class 0), and the class AA (which could be hazard).
   The addition of realizations of Z_j^C has – due to their construction – no influence on the rating class of a PD.

4. The larger the class interval, the more distinct the average reduction when using Z_j^C. Whereas, this tendency is perceptible, it distinctly depends on the curvature of K_j, which has a strong influence on the extent of the reduction.[37] An isolated consideration of the impact of the class interval on the expected reduction is therefore not possible in this context. Thus, the validity of this effect is only supposable. The exceptions of the mentioned effect can bee seen in class 1,2, and 6.

5. The reduction within each class never reaches the maximum expected reduction. This is due to the choice of the distributions for Z_j^O and Z_j^C which blur the probability mass within the according range.

4.3 Effects for homogeneous portfolios

Simulation design.
In this setup, we examine the reduction effect for homogeneous credit portfolios and the possible structural change of those portfolios, as well. Following the pattern of Jankowitsch *et al.* (2003), we create four credit portfolios by

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[37] This can be realized by an Taylor expansion argument.

using different beta distributions.[38] For parameterizing the distributions, we choose $A = 0$ and $B = 1$, which ensures $PD_j \in [0, 1]$. The parameters α and β specify the shape of the distribution. By considering different pairs (α, β) of these parameters, we are able to model different kinds of quality of the credit portfolios:[39]

The proceeding is in all four cases the same: In a first step we simulate a portfolio by drawing 10,000 realizations of the according beta distribution for the PDs. For the further examination we fix the PDs in each portfolio and determine the overall capital charge. In a second step we draw for each loan in the portfolio one realization of a uniformly distributed random variable, which we fix for the further consideration of this portfolio, too. The realizations of these random variables are thereafter transformed on the one hand to realizations z_j^O of Z_j^O and on the other hand to realizations z_j^C of Z_j^C.[40] After the addition of those realizations the capital requirements of the modified portfolio are determined and compared to the original ones. The structure of the modified portfolio is compared to the original one, as well. However, we examine the portfolio's structure before and after adding realizations of white noise random variables only with respect to Z_j^O since the addition of Z_j^C (according to its definition) does not change the structure.

We do not determine the results of the total reduction in the entire portfolio but the results of the *average* reduction when dividing by the number of loans. This means an abstraction from the particular number of loans (and is admissible since we set $EAD_j = 1$ for all loans j, which additionally yields $CR_j = K_j$ for all j).[41]

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[38] The analytical properties and skewness of these distributions are a motivation for this proceeding. Furthermore, Renault and Scaillet (2003) use beta distributions to model recovery rates, see Jankowitsch *et al.* (2003).

[39] For an illustration of the according density functions, see Jankowitsch *et al.* (2003). There, the denotations p and q are used for α and β.

[40] These values are perfectly correlated, again. For the underlying idea, see the corresponding footnote in Section 4.2.

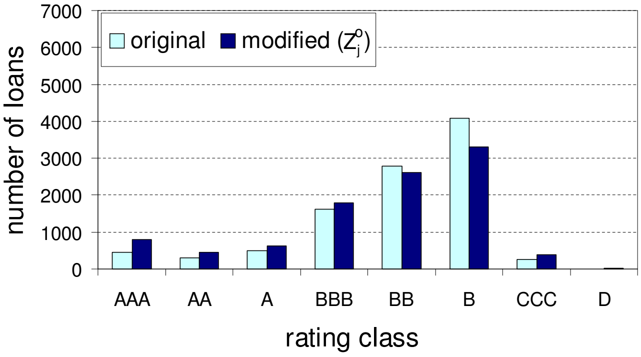
[41] To get the *total* capital requirements and reduction of a homogeneous portfolio with

Results.

The simulations yield the following results:[42]

- Portfolio 1: $\alpha = 0.7,\ \beta = 37.6$.

  This portfolio represents a concrete portfolio, see Jankowitsch *et al.* (2003). It is quite good.



  We observe the following reduction of the capital charges:

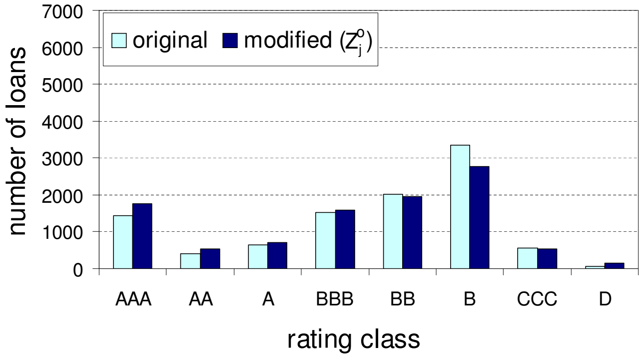
	original	using Z_j^O	change (Z_j^O)	using Z_j^C	change (Z_j^C)
avg. K(PD)	7.279 %	6.768 %	-0.511 %	7.265 %	-0.014 %
avg. class	4.93	4.66	-0.27	4.93	0

- Portfolio 2: $\alpha = 0.4,\ \beta = 19$.

  Compared to portfolio 1 there are less loans in the rating classes BBB to B and more loans in both the best classes AAA to A and the worst classes CCC and DD. Nevertheless, we consider this portfolio better than portfolio 1.

$EAD_j = const$ for all j, respectively, the results simply need to be multiplied with 10,000 $\cdot$ $const$.

[42] The changes of the capital requirements are specified as absolute changes, again. The exact portfolio structures are listed in Appendix A.2.

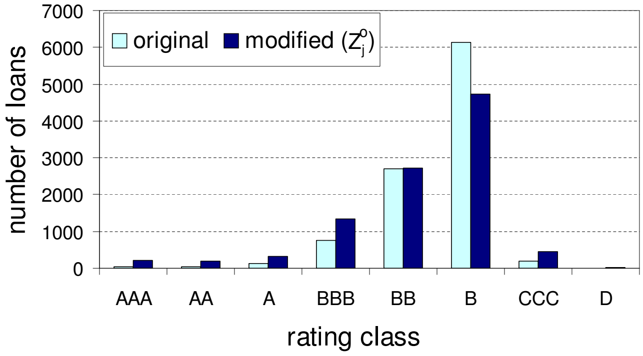


The observed reduction in portfolio 2 is:

	original	using Z_j^O	change (Z_j^O)	using Z_j^C	change (Z_j^C)
avg. K(PD)	6.740 %	6.208 %	-0.532 %	6.696 %	-0.044 %
avg. class	4.48	4.26	-0.22	4.48	0

- Portfolio 3: $\alpha = 1.4,\ \beta = 58$.

  In contrast, this portfolio is rather bad.

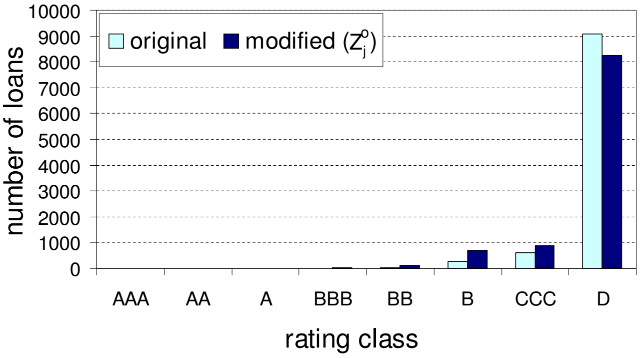


In portfolio 3, the determined capital charges are:

	original	using Z_j^O	change (Z_j^O)	using Z_j^C	change (Z_j^C)
avg. K(PD)	8.775 %	8.125 %	-0.650 %	8.757 %	-0.018 %
avg. class	5.53	5.22	-0.30	5.53	0

- Portfolio 4: $\alpha = 1.5,\ \beta = 1.3$.

  This is an extremely bad portfolio (which should not be considered realistic, actually, but is interesting for assessing the reduction's extent).



Finally, the capital requirements of portfolio 4 are the following:

	original	using Z_j^O	change (Z_j^O)	using Z_j^C	change (Z_j^C)
avg. K(PD)	13.814 %	12.235 %	-1.579 %	13.692 %	-0.122 %
avg. class	7.87	7.71	-0.16	7.87	0

Interpretation.

Again, the reduction effect is distinct and leads to a clear diminution of the capital requirements. It reaches from a reduction of 51 bp in portfolio 1 to 158 bp in portfolio 4 when using Z_j^O, and a reduction of 1 bp in portfolio 1 to 12 bp in portfolio 4 when using Z_j^C. When considering real values of the EADs, this a vast reduction of the capital charges.

The effects 1, 2, and 3 that are observable for single loans hold in the portfolio context, as well. Effect 1 is reflected in the different reduction extents of the portfolios: The worse the portfolio, the larger the reduction. The only exception is portfolio 2 which is actually considered better than portfolio 1 but shows a larger reduction of the requirements than portfolio 1 (which is probably caused by the larger number of loans in the worst classes). Effect 2 holds exactly for the portfolio context, i.e. the addition of realizations of Z_j^O leads to larger reductions than the addition of realizations of Z_j^C. Finally, effect 3 can be seen in the structural change of the portfolio.

Note that the results in this setup are stochastic since we consider *one* realization of an addition of white noise to the portfolio. Indeed, when passing further simulations the deviation of the average reduction could come to ± 5 bp, roughly, when adding realizations of Z_j^O.

However, our examination yields an assessment of the reduction's extent in a homogeneous portfolio context. To give a numerical example: When considering a total EAD volume of € 2.5 bn[43], i.e. each EAD equals € 250,000, the original capital charge in portfolio 1 is about € 182 m. The reduction when modifying the PDs amounts to roughly € 13 m (using Z_j^O) and € 0.35 m (using Z_j^C), respectively.

4.4 Effects for heterogeneous portfolios

Simulation design.

To get an impression of the reduction's extent in more realistic scenarios we now examine the described effect in the context of heterogeneous credit portfolios. For comparison purposes we use the fixed credit portfolios 1 to 4 of the previous section but now consider different exposures at default. While using the values $LGD_j = 45\%$, $M_j = 2.5$, and $S_j = 50$ (€ m) for all j, again, we create random EAD values by drawing 10,000 realizations of a lognormal distribution with the parameters $\mu = 12$ and $\sigma = 0.9265$ (implying a mean of 250,000 and standard deviation of 291,490). This *skewed* distribution is chosen with regard to the intuition that small and medium EADs occur with higher probabilities than large EADs.

Then, the 10,000 obtained EADs and 10,000 PDs in each portfolio are assigned to each other at random.[44] The used realizations of the EADs are fixed

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[43] which can be considered realistic in view of balance sheet data of banks. The loan volume of *big banks* can exceed this value many times, see e.g. Bayerische Hypo- und Vereinsbank AG (2005).

[44] The rationale is that we see no intention why borrowers with high exposures should have systematically lower and higher PDs, respectively.

(i.e. we use the same EADs with identical frequencies in all four portfolios) in order to provide better comparison possibilities. Moreover, the EADs are rounded up to the next multiple of € 10,000 because we assume that borrowers do not take loans of arbitrary amounts.

The obtained set of the drawn EADs has the following properties:[45]

- The 10,000 EADs in total sum up to an amount of about € 2.5 bn.
- The maximum EAD is € 3.86 m.
- The (empirical) 99 %-quantile of the EADs is about € 1.45 m, i.e. 100 of the 10,000 loans have an EAD exceeding this amount.
- The 100 biggest loans (with respect to their EAD) have a total EAD of roughly € 201 m.

After assigning these 10,000 EADs to the PDs in each portfolio the total capital charges CR for the four portfolios are determined.

To quantify the reduction effect we consider two setups: Firstly, we perform exactly the same procedure as in the previous section, i.e. we add those realizations of white noise random variables to the PDs that we already used in the previous section (this is due to comparison purposes). This is done both for Z_j^O and Z_j^C. Thereafter, we compute the total capital requirements in both cases and compare them to the original ones.

Secondly, we add realizations of the white noise random variables only to the "big fishes", i.e. the PDs that are associated with the 100 highest exposures. Once again, we compare the obtained total capital charges to the original ones. The aim of this proceeding is to analyze whether focussing on a modification solely of the *big loans* is worthwhile with regard to the requirements' reduction.

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[45] The exact values and a diagram showing the amounts and according frequencies of the created EADs is given in Appendix A.2.

Results.

With the denotations CR^O for the total capital requirements of the portfolio modified by using Z_j^O for all j, ΔCR^O for the total change in the capital charges, i.e. $CR^O - CR$, and CR^C and ΔCR^C for the according values when using Z_j^C for all j, we obtain:[46]

- Portfolio 1:

  The total capital requirements of the original credit portfolio amount to € 185,218,288.

  Whereas, we obtain the following results when modifying the PDs:

modified PDs	CR^O	ΔCR^O	CR^C	ΔCR^C
all	172,207,623	-13,010,665	184,858,905	-359,383
100 biggest loans	184,324,466	-893,821	185,199,134	-19,153

- Portfolio 2:

  The original credit portfolio has total capital charges of € 169,714,445.

  The modified PDs yield:

modified PDs	CR^O	ΔCR^O	CR^C	ΔCR^C
all	156,311,064	-13,403,381	168,466,249	-1,248,196
100 biggest loans	168,354,656	-1,359,789	169,572,375	-142,071

- Portfolio 3:

  In this portfolio we have CR = € 221,878,168.

  In contrast, we observe the following capital charges when modifying the PDs:

modified PDs	CR^O	ΔCR^O	CR^C	ΔCR^C
all	206,859,744	-15,018,424	221,441,990	-436,178
100 biggest loans	220,757,707	-1,120,460	221,803,075	-75,093

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[46] In the following tables, the values of the capital requirements and the according changes are specified in €.

- Portfolio 4:

  The total capital charge of the original credit portfolio is € 349,184,928. The results when modifying the PDs are:

modified PDs	CR^O	ΔCR^O	CR^C	ΔCR^C
all	311,331,816	-37,853,112	346,327,620	-2,857,308
100 biggest loans	346,474,921	-2,710,008	348,748,593	-436,336

Interpretation.

Effect 1 of the previous section is reflected in this context, as well: The worse the portfolio, the more distinct the reduction. This holds both for the modification of all and the 100 biggest loans. However, portfolio 2 is an exception because it is considered better than portfolio 1 but shows larger reductions (which is, again, probably caused by the larger number of loans in the worst rating classes). Effect 2, i.e. the fact that Z_j^O leads to larger reductions than Z_j^C, holds in this context, as well. Since we used exactly the same realizations of the white noise random variables as in the context of homogeneous portfolios there is no difference between the portfolios' structures in the previous section and this scenario. Therefore, effect 3 holds, too.

We are able to compare the results of a modification of all PDs to those of a modification of the 100 biggest loans' PDs, as well: Although the reduction's extent is explicitly less when modifying the PDs of the 100 biggest loans it is distinct. The reduction when modifying *all* PDs reaches from roughly € 13 m to € 38 m (Z_j^O) and € 0.36 m to € 2.9 m (Z_j^C), respectively. In contrast, it reaches from about € 0.89 m to € 2.71 m (Z_j^O) and € 0.02 m to € 0.44 m (Z_j^C), respectively, when modifying the PDs of the *100 biggest loans*. However, except for portfolio 4, modifying the 100 biggest loans by Z_j^O leads to a larger reduction than when modifying all PDs by Z_j^C.

To sum up, there is (in most cases) a clear diminution of the capital requirements when using the modified PDs for determining the capital requirements.

Again, these results are obtained by a singular modification of each portfolio. Of course they are stochastic and therefore far from being universal. But they allow us to assess the extent of the reduction effect in the context of heterogeneous credit portfolios.

On the basis of the numerical results shown in this and the previous sections banks might have an interest in modifying their PD estimations in order to make use of the reduction effect. Next, we will therefore examine whether such a modification conforms to the Basel II guidelines concerning internal rating models.

5 Limitations

5.1 General requirements according to Basel II

In the following section we will examine the Basel II guidelines concerning an intentional modification of the rating system. The incentive not to improve an existing rating model is very important and should be alarming on the part of the supervisors. However, it is not being examined in this context because it is unlikely that the supervisors deny a model that has once been accepted for usage in the IRBA but has not been improved ever since. Therefore, this incentive is neglected in the following section.

The Basel Committee on Banking Supervision has formulated several requirements concerning rating models used in the IRB approach.[47] We give a short overview over those criteria which are relevant for the analysis if an estimator's modification, as described in the previous sections, is admissible.

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[47] See particularly Basel Committee on Banking Supervision (2006a), para. 387 - 537, 765 - 777, and 825 f..

1. Quantitative criteria.

The model has to be *accurate on average* and must not have any known material *biases*. It also has to be *stable* and should be able to provide a meaningful *differentiation of risk* and reasonably *accurate* and *consistent* quantitative estimates of risk.[48] Moreover, it must incorporate *business cycles* and also be *conservative*. The model is required to have *sound model relationships*.

2. Qualitative criteria.

The rating model must be *sound* in the sense of ranking order and quantifying risk in a consistent, reliable, and valid fashion. Furthermore, the used rating *criteria* should be *consistent* with the majority of the supervisory criteria.

Rating assignments based on *subjective judgement* are emphatically allowed but have to be *documented and validated* like the entire rating model, as well. The aligned ratings used in the IRB approach must play an essential role in the *credit approval, risk management, capital allocation, and corporate governance.*

Furthermore, there are some requirements concerning the *validation* of a rating model. The validation process is primarily to be performed and clearly documented by the bank *itself* since it is the bank's responsibility to prove the model's suitability. For this purpose the bank may use its own test methods. However, *stresstests and backtests* have to be passed *frequently* and at least once a year. The Supervisors may charge external reviewers with the validation of the model, as well, but usually review the bank's validation *results* and *documentation*. In case of the realizations continuing to be higher than expected values, the model's outcomes have to be adjusted upwards.

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[48] Note the subjectivity of the term "reasonably".

5.2 Getting a modification accepted

As seen in Section 3.2, banks might have an interest in adding certain white noise random values to their PD estimations and by this means reduce their expected capital charges. We now examine whether such a modification conforms to the Basel II guidelines and, if not, show possibilities for the bank to hide it.

1. Quantitative criteria:

A modified model resulting from an addition of the aforementioned white noise random variable to the PD estimators does not meet all quantitative criteria concerning rating models in the IRBA. Assume the *original* model fulfills the formulated quantitative criteria then the accuracy on average and the absence of biases in the modified model is a consequence of Lemma 1. The incorporation of business cycles, the property of estimating in a conservative way, and the model's relationships are neither affected on average. Thus, neither stresstests nor backtests should be able to reveal such a modification if the original model passes.

But a modified model is supposed not to meet the criterion of stability. It is also supposed to violate the criterion of consistent estimations of the default probabilities. Thereby, we consider the quantitative criteria not fulfilled by a modified model.[49]

2. Qualitative criteria.

Although the consistence with the majority of the supervisory criteria is not affected by adding white noise to the resulting PD a model worsened for the purpose of capital requirements' reduction should not be considered sound

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[49] Note that in Basel II no quantitative values are given as a benchmark for the quantitative criteria. Therefore, these properties need to be validated with respect to tangible guidelines of the national implementations of Basel II.

by the regulators in the aforementioned sense. Thus, the qualitative criteria are not fulfilled by a modified model, neither.

Subsequently, if banks intend to make use of the reduction effect they might try to manipulate their models in a manner described above and then hide this modification.

Since the regulators mainly rely on the *banks' internal validation process* to verify the rating systems there might be indeed suitable opportunities for the banks to do so: They could formulate the model's documentation in a manner that the supervisors do not get suspicious or at least do not have anything to object to. If the supervisors neglect tests concerning the model's stability there is a chance that the modification will not become visible.

Furthermore, there is even a possibility for the banks to succeed with a modification without hiding it: The supervisors concede the banks the opportunity of including subjective views into their rating process. This provides the banks exactly what they need to get the modification accepted. They could add a realization of the used white noise random variable to the original default probability and communicate the deviation as a subjective judgement. Of course, large and frequent deviations, respectively, need to be explained in a more convincing way. Therefore, either the (conditional) variance of the used white noise random variable should not be too large[50] or the banks should only modify the PDs of the biggest loans, which is worthwhile, as seen in Section 4.4.

Additionally, the structure of the credit portfolio can change vastly by applying unsuitable white noise (without any migrations to or from other banks) so that the supervisors might get suspicious, as well. But this is on the one hand an effect preventable by choosing suitable white noise random variables

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[50] This diminishes the reduction effect – but the acceptance of the model is the main constraint for the banks.

and on the other hand only a singular effect appearing in the very moment when the banks switch from the original to the modified model. Hereafter, the PD estimators will be following the distribution of X_j and a similar structural change will not occur again (neglecting “natural” changes in the distribution resulting from adjustments of the original model and migrations of the borrowers). It means a lot of work to proceed with *all* borrowers in this way, though. Banks should therefore choose some suitable borrowers for PD modifications, i.e. borrowers with high exposures, for making good use of the reduction effect.

However, the aforementioned opportunity of modifying the rating model also endangers the banks because the model has to be applied to credit approval, risk management, capital allocation, and corporate governance, too. Banks should therefore not only ogle at a possible reduction of capital requirements but also think of the perils that are correlated to such a modification.

5.3 Revealing a modification

Since a model worsened for the purpose of capital reduction in the aforementioned way should be denied by the supervisors (for reasons shown above) and since banks might try to hide such a modification we offer some advice which is intended to provide the supervisors possibilities to reveal a modification.

Firstly, the supervisors should test the model’s stability since a modified model does not meet this criterion.

Secondly, they should have a particular look at all ratings that are influenced by subjective judgements because this is a possibility for banks to get the modifications accepted. Hence, the reasons for deviations caused by subjective assessments should be examined. This is particularly important concerning big loans since they provide the highest reduction potential.

Thirdly, unexpected changes in the structure of the credit portfolio, namely migrations to “outer” classes, are an indicator for a possible modification by adding white noise to the ratings, too.
Fourthly, when the banks modify their PD estimations by adding white noise, oscillating PDs occur in the court of time. This is therefore an indicative of the attempt of making use of the reduction effect.

As soon as any sign of a possible modification is observed, regulators should initiate an inspection of the rating model, which may ultimately lead to the model’s rejection.

6 Summary and Perspectives

Starting point of our examination was the fact that the capital requirements for credit risk in Basel II’s IRB approach are a concave function of the PD. Therefore, more white noise in the PD estimator means lower capital charges. This reduction effect was described and proven mathematically in Section 3.1. The effect generates an incentive for banks not to improve or even intentionally worsen their rating system. Ways to do so were shown in Section 3.3 and quantified by simulations in Section 4. The reduction effect turned out to be (dependent on the particular PD which is considered) very distinct. Expected reductions of the capital requirement factor from 1 bp up to 189 bp were observed when modifying several PDs from 0.015 % to nearly 60 %. We also modified four credit portfolios, on the one hand under the assumption of homogeneity and on the other hand on the assumption of heterogeneity. In all portfolio considerations the reduction effect was very distinct, too, and led to vast diminutions of the total capital requirements. For example, in a heterogeneous portfolio with 10,000 loans and a total amount of about € 2.5 bn of exposure at default, the reduction of the capital total charges came to about € 13 m in a good portfolio, and about € 38 m in an extremely bad portfolio when using certain white noise random variables and modifying all PD estimations. When modifying only the PDs of the 100 biggest loans using the same random variables, the reduction's extent reached from € 0.89 m in a good portfolio to € 2.71 m in an extremely bad portfolio.

In Section 5 we examined supervisory limitations concerning the reduction effect. It turned out that a modified model fails to fulfill both the quantitative and qualitative criteria. We deduced two possibilities for banks to hide such a modification: On the one hand, as the supervisors base their validation mainly on the results of the banks' validation process, there might be possibilities for banks to palliate the documentation and thereby hide the modification. On the other hand, there is the possibility of communicating the deviations obtained by adding white noise as subjective judgements. However, the latter possibility is limited since it is cumbersome to apply such a procedure to each PD in the credit portfolio. Furthermore, the modified model has to be applied to corporate governance, which involves perils for the banks, as well. The only possibilities for the supervisors to reveal such a modification are to test the model's stability, to check the reasons for subjective judgements accurately, and to pay attention to structural changes of the portfolio and oscillating PDs in the court of time.

Nevertheless, banks might have an incentive not to improve their PD estimations – which is alarming and unintended by the Basel Committee.

A Appendix

A.1 Correlation, maturity adjustment and maturity in the IRBA

The correlation for corporate, banks and sovereign exposures in the IRB approach is determined by

$$\begin{aligned}\rho(PD_j) \quad = \quad & 0.12 \cdot \frac{1-e^{-50 \cdot PD_j}}{1-e^{-50}} + 0.24 \cdot \left(1 - \frac{1-e^{-50 \cdot PD_j}}{1-e^{-50}}\right) \\ & \underbrace{-0.04 \cdot \left(1 - \frac{S_j - 5}{45}\right)}_{\text{only for corporate}},\end{aligned}$$

where S_j denotes the annual turnover in € m $(5 \leq S_j \leq 50)$.
In contrast, the correlation for retail exposures is defined by

$$\rho_j(PD_j) = \begin{cases} 0.15 & \text{residential mortgage,} \\ 0.4 & \text{qualifying revolving retail,} \\ 0.03 \cdot \frac{1-e^{-35 \cdot PD_j}}{1-e^{-35}} + 0.16 \cdot \left(\frac{1-e^{-35 \cdot PD_j}}{1-e^{-35}}\right) & \text{other retail.} \end{cases}$$

For all asset classes, the maturity adjustment $b(PD_j)$ is calculated by

$$b(PD_j) = [0.11852 - 0.05478 \cdot ln(PD_j)]^2$$

and the maturity itself by

$$M_j = \begin{cases} 2.5 & \text{basic IRBA,} \\ \sum_t t \cdot \frac{CF_{j,t}}{\sum_t CF_{j,t}} & \text{advanced IRBA,} \end{cases}$$

where $1 \leq M_j \leq 5$ for the advanced IRBA and $CF_{j,t}$ denotes the cash flow of loan j at moment t.

A.2 Portfolio structures used for the simulations

The four portfolios we created for our simulations show the following classification of loans to rating classes:

Portfolio 1:

class	AAA	AA	A	BBB	BB	B	CCC	D
original	450	301	489	1,625	2,781	4,081	267	6
after adding Z_j^O	798	452	616	1,798	2,617	3,295	396	28
after adding Z_j^C	450	301	489	1,625	2,781	4,081	267	6

Portfolio 2:

class	AAA	AA	A	BBB	BB	B	CCC	D
original	1,441	418	635	1,516	2,029	3,341	549	71
after adding Z_j^O	1,765	536	707	1,586	1,956	2,770	536	144
after adding Z_j^C	1,441	418	635	1,516	2,029	3,341	549	71

Portfolio 3:

class	AAA	AA	A	BBB	BB	B	CCC	D
original	34	36	125	766	2,709	6,130	199	1
after adding Z_j^O	220	203	314	1,344	2,720	4,722	453	24
after adding Z_j^C	34	36	125	766	2,709	6,130	199	1

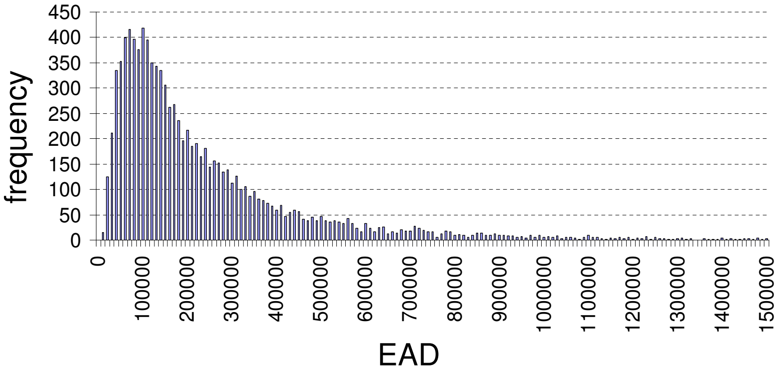
Portfolio 4:

class	AAA	AA	A	BBB	BB	B	CCC	D
original	0	0	1	3	22	278	615	9,081
after adding Z_j^O	3	2	13	37	124	691	876	8,254
after adding Z_j^C	0	0	1	3	22	278	615	9,081

For the examination of the heterogeneous portfolios, we draw 10,000 realizations of a lognormal distribution with the parameters $\mu = 12$ and $\sigma = 0.9265$ for modelling the EADs. The set of EADs has the following properties:

- All EADs are rounded up to the next multiple of € 10,000.
- The (empirical) mean of the EADs is € 247,906, the (empirical) standard deviation € 287,211.
- The 10,000 EADs in total sum up to an amount of € 2,528,770,000.
- The maximum EAD is € 3,860,000.
- The (empirical) 99 %-quantile of the EADs is € 1,450,100.
- The 100 largest loans have a total EAD of € 201,230,000.

The EAD amounts and according frequencies are:



In this diagram, the right tail is clipped in order to provide an acceptable overview (the largest EAD is € 3.86 m, and there is less than 100 EADs exceeding € 1.5 m).

References

Basel Committee on Banking Supervision (1988). International Convergence of Capital Measurement and Capital Standards.

Basel Committee on Banking Supervision (1996). Amendment to the Capital Accord to Incorporate Market Risks.

Basel Committee on Banking Supervision (2001). The New Basel Capital Accord: An Explanatory Note.

Basel Committee on Banking Supervision (2006a). International Convergence of Capital Measurement and Capital Standards: A Revised Framework – Comprehensive Version.

Basel Committee on Banking Supervision (2006b). The IRB Use Test: Background and Implementation.

Bayerische Hypo- und Vereinsbank AG (2005). Annual Report 2005.

Billingsley, P. (1995). Probability and Measure. John Wiley & Sons, New York, 3rd edition.

Broeker, F. (2004). Auswirkungen der Qualität von Ratingsystemen auf die regulatorische Eigenkapitalbelastung. In O. Everling and K.-H. Goedeckemeyer, editors, *Bankenrating*, pages 487–506. Gabler, Wiesbaden.

Doob, J. L. (1984). Classical Potential Theory and its Probabilistic Counterpart. Springer, Berlin.

Duefinance (2007). Die Ratingstufen der großen Agenturen. *URL: http://www.duefinance.de/pdf/ratingstufen.pdf (02/28/2007).*

Jankowitsch, R., Pichler, S., and Schwaiger, W. S. A. (2003). Rating Granularity and Basel II Capital Requirements. *Working Paper*.

Jankowitsch, R., Pichler, S., and Schwaiger, W. S. A. (2007). Modelling the Economic Value of Credit Rating Systems. *Journal of Banking and Finance*, **31**, 181–198.

Kiefer, N. M. and Larson, C. E. (2004). Evaluating Design Choices in Economic Capital Modeling: A Loss Function Approach. In A. Dev, editor, *Economic Capital: A Practitioner Guide*, pages 315–328. Risk Books, London.

Levy, H. (1998). Stochastic Dominance. Investment Decision Making under Uncertainty. Kluwer Academic Publishers, Boston.

Rauhmeier, R. and Scheule, H. (2005). Eigenschaften von Ratings und ihre Auswirkung auf die Kapitalanforderung nach Basel II. *Deutsches Risk*, pages 34–40.

Renault, O. and Scaillet, O. (2003). On the Way to Recovery: A Nonparametric Bias Free Estimation of Recovery Rate Densities. *HEC Geneve: FAME Research Paper No. 83*.

Rothschild, M. and Stiglitz, J. E. (1970). Increasing Risk: I. A Definition. *Journal of Economic Theory*, **2**, 225–243.