Policy Evaluation and Endogenous Heterogeneity: Reviving the Representative Individual

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Abstract

Although the representative individual framework is frequently used for policy evaluation, it is subject to well-known problems, especially the non-existence of representative preferences. We advance a complementary framework that avoids some of these problems without requiring additional information. Our concept is based on endogenously derived distributional information and is applicable whenever heterogeneity results from individual decisions. It provides a consistent description of a policy's long-run welfare effects and shows that the policy influences not only the welfare of a representative individual but also "who" that individual is. This effect can change a model's policy recommendations substantially.

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1 Introduction

Economic concepts for evaluating public policy are usually based on assessing the policy's effects on the welfare of the affected individuals. Frequently, a policy is judged by the induced change in the welfare of a representative individual. This framework results in mathematically manageable models, which can be used to address complex policy settings and which have minimal data requirements.

But the model of the representative individual has been subject to an extensive and elaborate criticism that has culminated in A. Kirman's suggestion that "[...] the "representative" agent deserves a decent burial, [. . .]" (Kirman, 1992, p. 119). Indeed, Kirman's succinct diagnosis shows that the representative individual suffers from several fatal diseases: Non-existence, lack of normative validity, and proneness to oversimplification.

The most serious charge is that of non-existence. As shown by Muellbauer (1975) and Lau (1982), we cannot take for granted that aggregate behavior or collective welfare can be described by a model that does not include distributional information. Indeed, such a description necessitates that individual preferences admit an indirect utility function of the Gorman type (Mas-Colell *et al.*, 1995, Ch. 4), which is a rather restrictive requirement.

In addition, it is possible to construct examples in which a representative individual prefers a situation A to a situation B, whereas all individuals whose preferences have been used in the aggregation prefer B to A, see Jerison (1984) and Dow and Werlang (1988). Thus the concept of the representative individual might lack normative validity.[1]

Finally, using the model of the representative individual in a normative setting implies abstracting from heterogeneity, from the distributional effects of public policy, and from differences in individual responses to public policy. Therefore this model does not convey information that can be essential for political decisions.

The last point may be set aside as a necessary side-effect of abstract modeling approaches and the second point has, so far, only been shown in specific examples. But non-existence is clearly fatal. It seems difficult to envision a more substantial failure of a modeling approach than the non-existence of its central component. So given that Kirman's criticism dates back more than a decade, it seems prudent to assume that the representative individual has been laid to rest.

But the representative individual seems to enjoy a decent afterlife. Despite its methodological difficulties, the representative individual framework is still ubiq-

[1]This problem can be solved by considering special classes of distribution rules, see, e.g., Chipman and Moore (1979) and Jerison (1994). However, this comes at the price of ranking policies for a fictive distribution, which, in some applications, might be a questionable approach to policy evaluation.

uitous in many areas of economics. The main reason is most likely the lack of practical alternatives. Although multi-agent models are increasingly used for policy evaluation,[2] these models are still technically challenging to an extent that prohibits their universal utilization. In some cases, their complexity is reduced by assumptions on individual preferences that assure that computationally important variables, like relative prices, depend only on aggregate but not on distributional information.[3] But even then, multi-agent models require substantially more information than the representative individual framework, especially distributional information, which is often not easily available.

In this paper, we inquire whether the representative individual, or at least its essential parts, can be saved from its pending burial. In contrast to studies that aim at showing that the representative individual framework is useful as an approximation, like Krusell and Smith (1998), or that it can be used for particular preferences or market conditions, like Hylland and Zeckhauser (1979) or Christiansen (1981), we advance an explicit aggregation procedure that eliminates the existence problem. The idea is to include distributional information into the representative individual approach but to generate this information endogenously.

Our concept is similar to that used in Aoki (1996), Aoki (2002), and Dorofeenko and Shorish (2005), but combines the distributional approach advanced there with a conventional microeconomic model of individual behavior. Furthermore, we address a normative problem of policy evaluation, whereas these studies are concerned with the positive question of describing aggregate behavior.

We show that in the particular setting considered in our analysis, it is possible to calculate a consistent aggregate of social welfare from a single model of individual behavior without using additional distributional information. Furthermore, this approach cures not only the existence problem but also mends some aspects of the problem of oversimplification by generating detailed information about the distributional consequences of a policy.

The setting that we consider is an economy consisting of individuals that differ only w.r.t. characteristics that, in the long-run, are influenced by their decisions. The generic example is an economy with income or wealth heterogeneity that results from differences in saving behavior or human capital investment decisions. We show that in such a setting, the long-run distribution of individual characteristics is endogenously determined and can be calculated from a dynamic model of individual behavior. Given this distribution, we can consistently calculate any measure of social welfare that retains anonymity. Thus like the representative individual concept, our approach is based solely on a single model of individual behavior; only, the aggregation procedure is somewhat more sophisticated.

[2]See, e.g., Kaplow (1996), Sandmo (1998), Norman (2004), or Hellwig (2005).

[3]See, e.g., Krusell and Smith (1998), Krusell and Ríos-Rull (1999), Caselli and Ventura (2000).

But compared to the representative individual framework, our concept provides substantially differing policy recommendations. It shows that in the long-run, public policy does not only change the welfare of a "representative individual," it also changes "who" the representative individual is. More precisely, a policy influences not only the welfare of an individual with given characteristics but also the distribution of individual characteristics in the economy. Consequently, the individual that is "representative" for one policy is usually not "representative" for a different policy.

This effect is discussed in Kirman (1992) and analyzed for specific examples in Geweke (1985) and Kupiec and Sharpe (1991). But to our knowledge, there is no general modeling framework that allows to include it into policy analysis.

Although the representative individual framework can be used to depict the influence of policy variables on average characteristics, like average income, it does not suffice to account for this effect. In many cases, the response of an individual to a policy will depend on the individual's characteristics. This implies that policy variables do not only influence average characteristics (like average income) but also the higher moments of the distribution of individual characteristics, like the variance or the skewness of the income distribution. In such cases, the representative individual framework is inadequate for evaluating the long-run consequences of a policy.

This shows that arguments that support the use of the representative individual framework for policy evaluation, as in Hylland and Zeckhauser (1979) or Christiansen (1981), neglect a point that is important in a long-run perspective. Using a multi-agent framework, these studies demonstrate that for a given distribution of individual characteristics, like wealth, income, or human capital, it is feasible to depict aggregate behavior as well as the costs and benefits of public policies by a representative individual approach. But they fail to recognize that the distribution itself is changed by the policy in the long-run.

So although the representative individual framework may be appropriate for policy analysis in a short-run setting, it is not sufficient for a long-run analysis. The concept advanced in this paper can be seen as a complement to the representative agent framework that is especially suited for evaluating a policy's long-run consequences.

In the next section, we set up our model and derive our main results. In Section 3, we provide an extended example. Section 4 concludes.

2 Reviving the Representative Individual

We advance our concept of policy evaluation in several steps. First, we set up our basic model and show how distributional information can be endogenously calculated. Second, we derive a general characterization of an optimal policy. Finally, we compare the results of our approach to those derived from a representative individual framework for a special case.

2.1 The Modeling Concept

Consider the following stochastic intertemporal utility maximization of an individual:

$$V(w_0, z) := \max_{c(t)\in\mathcal{C}} \mathcal{E}\left(\int_0^\infty e^{-\varrho t} U(w(t), c(t), z)dt\right), \tag{1}$$

$$dw = h(w(t), c(t), z)dt + \sigma_0(w(t), c(t), z)d\varepsilon, \tag{2}$$

$$w(0) = w_0 \in \mathcal{W}. \tag{3}$$

In this model, $U(w, c, z)$ is the instantaneous utility function of the individual and depends on consumption $c \in \mathcal{C} \subseteq \mathbb{R}^n$, a constant policy variable $z \in \mathcal{Z} \subseteq R^q$, and (possibly) on a stock of assets $w \in \mathcal{W} \subseteq R$, like wealth, human capital, or durable goods. The parameter ϱ denotes the discount rate of the individual. The function $h(w, c, z)$ depicts the trend of the stock w, which is influenced by individual consumption c and the policy z. The change of the stock is also subject to stochastic shocks that are depicted by $\sigma_0(w, c, z)d\varepsilon$, where $\sigma_0(w, c, z)$ describes the influence of these shocks, which can be subject to individual behavior, and where ε is a standard Wiener process.

The value function $V(w_0, z)$ yields the maximal expected discounted utility that the individual can achieve for a given policy z and a given initial stock w_0.

Our intention is to use the information embedded in the above model of individual behavior to characterize social welfare. Therefore we do not specialize this model to a setting where an explicite solution can be derived, but only assume that such a solution exists and that it has some convenient properties.

A1 The problem (1)–(3) has a unique, time-independent feedback solution $c^*(w, z)$ that is twice differentiable w.r.t. $w, z \in \mathcal{W} \times \mathcal{Z}$.

A2 The functions $U(w, c, z)$, $h(w, c, z)$, and $\sigma_0(w, c, z)$ are twice differentiable w.r.t. $w, z \in \mathcal{W} \times \mathcal{Z}$.

Note that A1 and A2 together imply that $V(w_0, z)$ is a twice differentiable function of $w_0, z \in \mathcal{W} \times \mathcal{Z}$.

A class of models for which these assumptions hold are portfolio selection problems with HARA-type utility functions, see, for example, Merton (1971) or Cooper *et al.* (1995). Another class, which we will extensively use throughout the paper, are models with a quadratic objective function and a linear constraint (2).

We consider an economy with a fixed number of individuals in which the individuals are heterogenous w.r.t. w_0 and the shocks $d\varepsilon$. For example, individuals can differ w.r.t. their initial wealth or w.r.t. to their initial stock of human capital and are subject to differing shocks to the value of their portfolio of assets or to their specific knowledge. Let $f(w_0) : \mathcal{W} \to \mathbb{R}_+$ denote the density of the distribution of w_0 in the economy.

We are interested in characterizing an optimal policy z in the context of a long-run perspective. Due to the restriction to the long-run, we consider only constant values of z, that is, we do not attempt to derive a dynamic policy that leads to an optimal transition to the long-run optimum. For presentational ease, we use the following simple utilitarian construct of social welfare,[4] but we could use any criterion that is invariant w.r.t. permutations of the individuals (i.e., any anonymous measure of social welfare).[5]

$$W(z) := \int_{\mathcal{W}} V(w, z) f(w, z) dw. \tag{4}$$

This concept of social welfare is routinely applied in normative economics, albeit it presumes that utility is fully interpersonally comparable.

The problem in applying Eq. (4) is that in many cases, the distribution of w will not be known. Furthermore, this distribution has to be consistent with the dynamics of individual behavior as they result from (1)–(3). Often the latter point will imply that the distribution of w is a function of the policy z, which we have already indicated by the notation used in (4).

A possible way to solve these problems is to calculate the density $f(w, z)$ of this distribution from the model of individual behavior. For this calculation, we use the following assumptions.

A3 $f(w, z)$ is a twice differentiable function of $w, z \in \mathcal{W} \times \mathcal{Z}$.

A4 The stochastic shocks of different individuals are independent from each other.

These assumptions should be seen as an approximate description of real-world settings. Especially, A4 is restrictive in that it excludes economy-wide shocks. But there seem to be interesting applications in which it is met at least approximately. For example, individuals could differ w.r.t. their specific knowledge and thus their human capital might be subject to independent shocks.

[4]It can be interpreted either as the sum of the welfare of all individuals (in which case, $W(z)$, as defined below, has to be multiplied by the number of individuals) or as the expected welfare of a randomly chosen individual.

[5]Instead of the value function, we could also use the instantaneous utility function. But using the value function leads to a more self-contained model.

Given our assumptions, we can calculate the long-run distribution of the stocks w from the model (1)–(3).

Proposition 1. *Assume A1–A4. Then the long-run distribution of* w *is given by Wright's Equation*[6]

$$f^{eq}(w,z) = e^{\phi(w,z)} \frac{c_1(z) + c_2(z) \int_{w_0}^{w} e^{-\phi(\omega,z)} d\omega}{\sigma^2(w,z)}, \tag{5}$$

with

$$\phi(w,z) = 2 \int_{w_0}^{w} \frac{g(\omega,z)}{\sigma^2(w,z)} d\omega. \tag{6}$$

For a given boundary condition $f^{eq}(\bar{w},z) = \bar{f}(z)$ *with* $\bar{w} \in \mathcal{W}$*,* $f^{eq}(w,z)$ *is uniquely determined and independent from the initial distribution* $f(w_0)$*.*

Proof. A1 assures that a unique solution $c^*(w,z)$ of (1)–(3) exists. Given this solution, the evolution of the stock w of an individual can be described by the following stochastic process.

$$dw(t) = g(w,z)dt + \sigma(w,z)d\varepsilon, \tag{7}$$

where $g(w,z) := h(w, c^*(w,z), z)$ and $\sigma(w,z) := \sigma_0(w, c^*(w,z), z)$. For any initial distribution $f(w_0)$, we can calculate the evolution of the density $f(w,z,t)$ over time from (7) by using the following Fokker-Planck-Equation (also known as Kolmogorov forward equation), which formalizes the requirement that the number of individuals has to be preserved locally.[7]

$$\frac{\partial f(w,z,t)}{\partial t} = -\frac{\partial f(w,z,t)g(w,z)}{\partial w} + \frac{1}{2}\frac{\partial^2 f(w,z,t)\sigma^2(w,z)}{\partial w^2}. \tag{8}$$

The long-run distribution $f^{eq}(w,z)$ is the stationary solution of (8). Thus it is characterized by

$$\frac{\partial f^{eq}(w,z)g(w,z)}{\partial w} = \frac{1}{2}\frac{\partial^2 f^{eq}(w,z)\sigma^2(w,z)}{\partial w^2}. \tag{9}$$

As shown in Wright (1945), Eq. (5) is the solution of (9). Its existence and uniqueness, under the above boundary condition and the condition that $\int_{\mathcal{W}} f^{eq}(w,z)dw = 1$, follow from the Picard-Lindelöf Theorem. □

[6] This equation has been derived in the context of population genetics from a Fokker-Planck equation similar to Eq. (8) below in Wright (1945).

[7] See the appendix for a derivation of this equation.

Proposition 1 is the basis on which we will characterize an optimal policy. Its most important implication is that the dynamic model (1)–(3) suffices to determine the long-run distribution of w in the economy; apart from a boundary condition, no exogenous distributional information is necessary. Given that the model (1)–(3) also allows to characterize the welfare of an individual with a given stock w, we can thus consistently calculate any aggregate of social welfare that obeys the condition of anonymity solely on the basis of a model of individual behavior. Thus we maintain the informational basis of the representative individual framework but use a more sophisticated aggregation procedure to gain a consistent description of social welfare.

Furthermore, the distributional information allows to calculate not only measures of social welfare but also positive aggregates, like aggregate demand functions or total saving. So in this modeling context, the question whether a positive representative individual exists and whether its preferences are consistent with our notion of social welfare is not relevant. We can derive any information, positive or normative, without recourse to representative preferences.

Calculating the necessary distributional information becomes possible by using a dynamic model of individual behavior for the evaluation of a static policy. The information about intertemporal behavior, which is usually regarded as being irrelevant in a static setup, allows to characterize the steady-state distribution of individual characteristics. Note that by (5), the shape of the steady-state distribution is indeed determined by individual behavior and not by the stochastic shocks, which serve mainly as a stimulus for the continued existence of heterogeneity.

Eq. (8), which facilitates the aggregation from the microeconomic to the macroeconomic level, is frequently used in finance and has been employed in a way to which our approach comes close in Aoki (1996), Aoki (2002), and Dorofeenko and Shorish (2005). The main differences are that in these studies, the model is based on a set of discrete states and that in Aoki (1996), Aoki (2002), the transition probabilities are directly specified and not derived from a conventional microeconomic model. Also, all of these studies analyze aggregate behavior in a positive context, whereas we are concerned with the normative question of policy evaluation.

2.2 The Optimal Policy

Equation (5) generates endogenous distributional information from a dynamic model of individual behavior. We now use this information to evaluate the policy z. For this, we have to take into account that in most cases, the choice of this policy will be subject to constraints, like budget constraints or market clearing conditions. Therefore we introduce a set of constraints by demanding that

$$\int_{\mathcal{W}} y(w,z) f^{eq}(w,z) dw = x(z). \tag{10}$$

Here $y(w,z) \in \mathbb{R}^r$ describes individual behavior, like demand, saving, or tax payments. These quantities are usually determined by the solution of (1)–(3). The function $x(z) \in \mathbb{R}^r$ is used to introduce aggregate information into (10), like the supply of a private good or the costs of providing a public good. In most cases, this information has to be specified in addition to the model (1)–(3). We assume that $s > r$, that is, there have to be more policy variables than constraints, so that at least one degree of freedom remains in setting the policy.

Eq. (10) can depict budget constraints or similar limitations to public policy. It can also be used to describe price-based effects of a policy, like crowding-out effects or, more generally, policy induced changes in the demand or supply of private goods. For this, we interpret a part of the vector z as prices and let some of the constraints (10) describe market clearing conditions that determine these prices in dependency on the remaining policy variables. Given that the complete vector z is included in our model of individual behavior, this approach is sufficient to model most forms of market responses to public policy.

With (10), (5), and (4), we can characterize an optimal policy.

Proposition 2. *Candidates z for an inner maximum of* (4) *under the constraint* (10) *are characterized by*

$$\begin{aligned} &\int_{\mathcal{W}} \left(\frac{\partial V(w,z)}{\partial z} + V(w,z) \frac{\partial A(w,z)}{\partial z} \right) f^{eq}(w,z) dw \\ &= \lambda^T \int_{\mathcal{W}} \left(\frac{\partial y(w,z) - x(z)}{\partial z} + (y(w,z) - x(z)) \frac{\partial A(w,z)}{\partial z} \right) f^{eq}(w,z) dw, \end{aligned} \tag{11}$$

with

$$\begin{aligned} \frac{\partial A(w,z)}{\partial z} &= \frac{\partial \phi(w,z)}{\partial z} - \frac{2}{\sigma(w,z)} \frac{\partial \sigma(w,z)}{\partial z} \\ &+ \frac{\frac{\partial c_1(z)}{\partial z} + \frac{\partial c_2(z)}{\partial z} \int_{w_0}^{w} e^{-\phi(\omega,z)} d\omega - c_2(z) \int_{w_0}^{w} \frac{\partial \phi(\omega,z)}{\partial z} e^{-\phi(\omega,z)} d\omega}{c_1(z) + c_2(z) \int_{w_0}^{w} e^{-\phi(\omega,z)} d\omega}, \end{aligned} \tag{12}$$

with $f^{eq}(w,z)$ being given by (5)*, and with λ being the vector of Lagrange multipliers associated with* (10).

Proof. Maximizing (4) w.r.t. z under (10) and taking (5) into account directly yields the assertion. □

Proposition 2 shows that with an endogenously determined distribution, a policy has more effects than can be captured in the representative individual framework. The policy has not only a direct influence on the welfare of a given (and perhaps "representative") individual but also changes the distribution of individual characteristics in the economy; it influences both the welfare of the representative individual and "who" that representative individual is.

More formally, there are direct effects on individual welfare, as given by $\int_{\mathcal{W}}(\partial V(w,z)/\partial z)f^{eq}(w,z)dw$ on the l.h.s. of (11), and on the policy constraint, as depicted by $\int_{\mathcal{W}}(\partial(y(w,z)-x(z))/\partial z)f^{eq}(w,z)dw$ on the r.h.s. of (11). They describe the average marginal change of individual welfare and the marginal effect on the constraints due to the average policy-induced changes in individual behavior. These effects can be covered in the representative individual framework.

But as (11) shows, there are additional effects. They result from changes to the distribution of w in the economy caused by the policy via its effects on individual behavior. On the l.h.s. of (11), we have the term $\int_{\mathcal{W}}(V(w,z)\partial A(w,z)/\partial z)f^{eq}(w,z)dw$, which measures the policy-induced marginal change in the relative prevalence of individuals with a stock w weighed with the welfare of these individuals and averaged over $\mathcal{W}$. So this term can be seen as the marginal change in social welfare due to the distributional effects of z. On the r.h.s. of (11), we have a similar term that describes the marginal effect of a policy-induced distributional change on the constraints (10). Note that these effects are not simple redistributional effects but are due to changes in individual behavior caused by the policy.

To gain some intuition concerning the causes of these effects, it is helpful to consider an informal example.[8] Assume that the policy variables are the quantity of a public good and an income tax. There are two constraints: A budget constraint (tax revenue has to finance the expenditure for the public good) and a market clearing condition for a private good. The model (1)–(3) describes the decision of an individual between consumption and investment into human capital, the latter of which increases future income.

The policy will usually influence individual behavior, for example, a higher income tax might induce a reduction of the investment in human capital and an adjustment of the long-run level of consumption. Thereby, the policy will modify the income distribution. This will change the social costs of taxation, because individuals with different human capital will be affected differently by income taxation and thus suffer different welfare losses. Similarly, it will change the social benefits of public good provision, because individuals with different income will usually have different preferences between public and private goods. These points constitute the effects measured by $\int_{\mathcal{W}}(V(w,z)\partial A(w,z)/\partial z)f^{eq}(w,z)dw$ in (11).

The distributional changes can also influence the policy constraints. A change

[8] We will formalize a part of the following arguments in Section 3.

in the income distribution implies a change in the tax revenue and thus in the possible expenditure for the public good. It also implies a change in the demand for the private good, so that the price of the private good will adjust. These point are captured by $\int_{\mathcal{W}}(y(w,z)-x(z))\frac{\partial A(w,z)}{\partial z}f^{eq}(w,z)dw$ on the r.h.s. of (11).

Of course, some of these effects can be depicted with a conventional model of a representative individual by assuming that this individual's stock of human capital depends on the policy. However, apart from highly specialized setups, the effects of the policy on individual behavior will depend on individual characteristics; individuals will adjust their investment behavior differently to taxation depending on their human capital and income. In such cases, the policy will not only influence average characteristics but also the higher moments of the distribution of individual characteristics. The representative individual framework can only account for changes in the average characteristics. Therefore, it might miss important long-run effects of a policy. In the following section, we will elaborate on this point.

2.3 A Comparison with the Representative Individual for a Special Case

It is difficult to calculate the differences in policy recommendations between our concept and the representative individual framework in a general setup, because these differences depend on the details of the setup, that is, on the functional specification of the model (1)–(3). However, it is possible to characterize these differences in a special case that can be interpreted as an approximation to more general cases.

For this, we use the following quadratic approximation of the value function $V(w,z)$ of the individual optimization problem[9] (1)–(3)

$$\tilde{V}(w,z) \approx V_0 + V_w w + V_z z + \frac{1}{2}V_{ww}w^2 + V_{wz}wz + \frac{1}{2}V_{zz}z^2. \tag{13}$$

Here $V_w, V_z, V_{ww}, \ldots$ denote the first and second derivatives of V evaluated at $w = z = 0$ and V_0 denotes $V(0,0)$.

Furthermore, we assume that the functions in (2) have the following form

$$h(w,c,z) = h_0 + h_c c + h_{wz}wz, \tag{14}$$

$$\sigma_0(w,c,z) = \bar{\sigma}. \tag{15}$$

[9] Such a quadratic value function will, e.g., result from a quadratic approximation of the utility function together with a linear intertemporal constraint.

These settings characterize a simple case in which the effects of the policy z on individual behavior c depend on the individual characteristics w.

Finally, we assume that the utility function U does not depend on w and that $\mathcal{C} = \mathbb{R}$. In this case, the Hamilton-Jacobi-Bellman equation corresponding to (1)–(3) is

$$\varrho V(w,z) = \max_{c\in\mathbb{R}} \left[U(c,z) + \frac{\partial V(w,z)}{\partial w} \left(h_0 + h_c c + h_{wz} wz \right) + \frac{1}{2}\bar{\sigma}^2 \frac{\partial^2 V(w,z)}{\partial w^2} \right]. \tag{16}$$

Maximizing the r.h.s. of (16) w.r.t. c, we get

$$\frac{\partial U}{\partial c} = -h_c \frac{\partial V(w,z)}{\partial w} = -h_c \left(V_w + V_{ww} w + V_{wz} z \right). \tag{17}$$

Let $c^*(w,z)$ be the optimal value of c. Following the procedure used in Cooper *et al.* (1995), we substitute this optimal value into (16), differentiate w.r.t. w, and use (17). This leads to

$$c^*(w,z) = \frac{\varrho(V_w + V_{ww}w + V_{wz}z) - h_{wz}z(V_w + 2V_{ww}w + V_{wz}z) - h_0 V_{ww}}{h_c V_{ww}}, \tag{18}$$

$$g(w,z) := h(w, c^*(w,z), z) = \frac{(\varrho - h_{wz}z)(V_w + V_{ww}w + V_{wz}z)}{V_{ww}}, \tag{19}$$

$$\sigma(w,z) := \sigma_0(w, c^*(w,z), z) = \bar{\sigma}. \tag{20}$$

With this description of optimal individual behavior and under the assumptions that $\mathcal{W} = \mathbb{R}$ and that $z > \varrho/h_{wz}$, we can use Proposition 1 to calculate the resulting distribution of individual characteristics w:[10]

$$f^{eq}(w) = \frac{\sqrt{h_{wz}z - \varrho}}{\bar{\sigma}\sqrt{\pi}} e^{-\frac{(h_{wz}z-\varrho)(V_w + V_{ww}w + V_{wz}z)^2}{V_{ww}^2 \bar{\sigma}^2}}. \tag{21}$$

So in the steady-state, w is normally distributed with $\mathcal{E}(w) = -(V_W + V_{wz}z)/V_{WW}$ and $\mathrm{Var}(w) = \bar{\sigma}^2/(2(h_{wz}z - \varrho))$. Thus both the average value of w and the variance of w depend on the policy z.

With this information, we can calculate social welfare and the optimal policy. Substituting (21) and (13) into (4) yields

$$W(z) = V_0 - V_w \frac{V_w + V_{wz}z}{V_{ww}} + V_z z + V_{ww} \frac{\bar{\sigma}^2}{4(h_{wz}z - \varrho)} + V_{zz}\frac{z^2}{2} - V_{wz}z \frac{V_w + V_{wz}z}{V_{ww}}. \tag{22}$$

[10]Due to $\mathcal{W} = \mathbb{R}$, we use $\lim_{w\to\pm\infty} f(w,z) = 0$ as boundary conditions.

Assuming that there are no constraints for setting z, the optimal policy is thus characterized by[11]

$$V_z + V_{zz} z = V_{wz} \frac{V_w + 2V_{wz} z}{V_{ww}} + V_w \frac{V_{wz}}{V_{ww}} + V_{ww} \frac{h_{wz} \bar{\sigma}^2}{4(h_{wz} z - \varrho)^2}. \tag{23}$$

In contrast, in a representative individual framework, the measure for policy evaluation is given by

$$\hat{W}(z) = V_0 + V_w \hat{w} + V_z z + \frac{1}{2} V_{ww} \hat{w}^2 + V_{wz} \hat{w} z + \frac{1}{2} V_{zz} z^2, \tag{24}$$

for suitably defined "representative" characteristics $\hat{w}$. The simplest possible concept is to use constant characteristics $\hat{w}$, which yields the following characterization of an optimal policy:

$$V_z + V_{zz} z = -V_{wz} \hat{w}. \tag{25}$$

A more sophisticated application of the representative individual framework calculates the expected characteristics for a given policy z from (2), which results in $\hat{w} = -(V_w + V_{wz} z)/V_{ww}$. Note that this equals the expected characteristics calculated via Proposition 1. In this approach, the measure for evaluating the policy z is given by

$$\hat{W}(z) = V_0 - V_w \frac{V_w + V_{wz} z}{V_{ww}} + V_z z + V_{ww} \frac{(V_w + V_{wz} z)^2}{2V_{ww}} + V_{zz} \frac{z^2}{2} - V_{wz} z \frac{V_w + V_{wz} z}{V_{ww}}. \tag{26}$$

Consequently, the condition for an optimal policy is

$$V_z + V_{zz} z = V_{wz} \frac{V_w + 2V_{wz} z}{V_{ww}} + V_w \frac{V_{wz}}{V_{ww}} - V_{wz} \frac{V_w + V_{wz} z}{V_{ww}}. \tag{27}$$

A comparison of (23) with (25) and (27) shows that even in this simple quadratic setup without policy constraints, the representative individual framework, in both its simple and its sophisticated usage, misses effects of the policy z.

The simplistic approach leading to (25) neglects all effects that result from adjustments of individual behavior to the policy. Thus this approach may be appropriate for an analysis of the short-run but not of the long-run effects of the policy.

The more sophisticated representative individual concept includes effects that result from the adjustment of individual behavior to policy changes. But it depicts these effects wrongly. Although the first two terms on the r.h.s. of (23) and (27)

[11]Constraints on z of the type specified in (10) can be easily included and result in further differences between the results of our approach and those of the representative individual framework.

coincide, the third terms on the r.h.s of these equations differ substantially. The reason is that the representative individual framework cannot describe effects of the policy on the distribution beyond changes of the expected value of w and that even in the simple setup considered here, higher order effects are relevant. It uses $\mathcal{E}\left(w\right)^2$ where a higher order term (Var (w)) is needed for a consistent description of the policy effects and thus it yields incorrect recommendations.

In this simple example, the characteristics of the "real" representative individual can be calculated. They are given by

$$\hat{w} = -\frac{V_w + V_{wz}z}{V_{ww}} \pm \frac{\sqrt{(h_{wz}z - \varrho)(V_{ww}^2\bar{\sigma}^2 - 2(V_w + V_{wz}z)^2(h_{wz} - \varrho))}}{V_{ww}(h_{wz}z - \varrho)\sqrt{2}}. \quad (28)$$

As this expression shows, the adequate representative characteristics equal the above used characteristics of the conventional representative individual plus a term that corrects for the effects of the policy on the higher moments of the distribution. This correction term cannot be derived directly from (1)–(3). Thus to know "who" the individual is for a given policy, we have to calculate the distribution of individual characteristics in dependency on the policy. This is what our approach does.

3 An Extended Example

As an example, we apply our approach to a problem of public good provision and human capital investment. The intentions of this example are to show how our approach can be applied to a standard problem of policy evaluation and to compare its results to those gained from the representative individual framework.

To facilitate a clear exposition that remains focused on our approach and not on the particularities of the chosen setup, we keep the example as simple as possible. Especially, we use a partial equilibriums setting, that is, we take relative prices as given and fixed.

We first derive the optimal policy from our approach and then compare it to the results of the representative individual concept in a numerical study. Finally, we show how our approach can be extended to cover exogenous heterogeneity.

3.1 Optimal Taxation and Public Good Provision

Consider individuals that differ w.r.t. their stock of human capital w and whose income is proportional to this stock. The individuals derive utility from the consumption c of a private good as well as from the availability of a public good. For simplicity, we assume that the marginal utility of the public good is constant.[12]

The individual stocks of human capital can be increased by investment. They are subject to stochastic shocks that model unpredictable changes of the value of individual knowledge, for example due to technological progress or due to occupational changes. We assume that the standard deviation of these changes increases linearly with the stock of human capital.

The policy variables are a proportional income tax τ and the provided quantity Z of the public good. The total expenditure for the public good has to equal the revenue from the income tax.

To solve the model analytically, we approximate the utility function of the individual with a quadratic function. Furthermore, we assume that the individuals discount future welfare at a constant rate ϱ. With these settings, the consumption-vs-investment decision problem of the individual is given by

$$V(w_0, \tau, Z) = \max_{c \in \mathbb{R}} \int_0^\infty e^{-\varrho t} \left(c - \alpha c^2 + \beta Z\right) dt, \tag{29}$$

$$dw = (p(1-\tau)w - c)dt + \sigma w d\varepsilon, \tag{30}$$

$$w(t) \geq 0, \ \forall t > 0, \tag{31}$$

$$w(0) = w_0 \geq 0. \tag{32}$$

Here p denotes the wage rate w.r.t. effective labor supply, and $p(1-\tau)w - c$ is the human capital investment of the individual. In line with our assumptions of Section 2, we assume that ε is a standard Wiener process.

Note that the quadratic utility function should be seen as being only an approximation, because for high levels of consumption, the marginal utility w.r.t. c becomes negative. Furthermore to facilitate an explicite solution, we have allowed for $c < 0$ in (29). We will assess the relevance of both points in detail below.

The solution of (29)–(32) is given by

$$c^*(w, \tau) = w(2p(1-\tau) + \sigma^2 - \varrho) - \frac{p(1-\tau) - \varrho}{2\alpha(p(1-\tau) + \sigma^2)}. \tag{33}$$

So the assumptions A1 and A2 are met. The value function of the individual optimization problem is

$$\begin{aligned} V(w_0, \tau, Z) =& 2pw_0^2\tau\alpha + \frac{1}{4\varrho\alpha} + \frac{Z\beta + w_0\varrho\left(w_0\alpha\left(\varrho - \sigma^2 - 2p\right) + 2\right)}{\varrho} \\ &+ \frac{\left(2w_0\alpha\varrho + 1\right)\left(\sigma^2 + \varrho\right)}{2\alpha(p\varrho\tau - \varrho\left(\sigma^2 + p\right))} + \frac{\left(\sigma^2 + \varrho\right)^2}{4\alpha(\varrho\left(\sigma^2 + p - p\tau\right)^2)}. \end{aligned} \tag{34}$$

[12]We could allow for a declining marginal utility of the public good. This would change the numerical values calculated below but not our general conclusions.

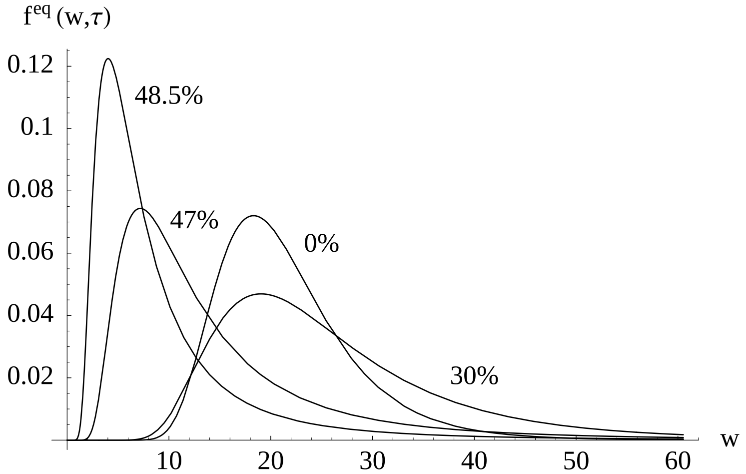

Figure 1: The density of the human capital distribution for different values of the income tax τ.

Thus, as is well known for linear-quadratic problems, the value function is a quadratic function of the stock of human capital.

Let us further assume that the distribution of human capital has a twice differentiable density $f(w, t, \tau, Z)$, and that the stochastic changes of the stocks of the individuals are independent from each other. Then we can apply Proposition 1 to gain information concerning the long-run distribution of human capital that is consistent with the model (29)–(32) of individual behavior.

Under the assumption that $p(1-\tau) > \varrho$ and the boundary condition $f^{eq}(0, \tau, Z) = 0$, which assures that all individuals have non-empty budget sets, the steady-state distribution of human capital in the economy is given by the following distribution.

$$f^{eq}(w,\tau) = \frac{e^{-\frac{\varrho+p(\tau-1)}{w\sigma^2(2p\alpha\tau-2\alpha(\sigma^2+p))}} w^{\frac{\varrho+p(\tau-1)}{\sigma^2}-3} \left(\frac{\varrho+p(\tau-1)}{\sigma^2(2p\alpha\tau-2\alpha(\sigma^2+p))}\right)^{\frac{2\sigma^2+p-\varrho-p\tau}{\sigma^2}}}{\Gamma\left(\frac{2\sigma^2+p-\varrho-p\tau}{\sigma^2}\right)}. \quad (35)$$

We have depicted this distribution in Figure 1 for the following parameter values: $\varrho = 0.1, p = 0.2, \alpha = 0.1, \sigma = 0.1$. For these values, the assumption $p(1-\tau) > \varrho$ implies feasible tax rates between 0 and 50%.[13]

The figure shows that income taxation has a substantial influence on the distribution of human capital in this example. For a low tax rate, the distribution is

[13]For higher tax rates, it is not individually optimal to accumulate human capital, so that the density degenerates to a Dirac function at $w = 0$.

only slightly skewed and has a comparatively small variance. There are few people with exceptionally high but also few people with very low human capital.Taxation increases the skewness of the distribution. Up to intermediate tax levels, it also increases the expected human capital and the variance of human capital. But if the tax rate exceeds a certain level (about 35% for the above parameter values), the expected human capital stock and the variance decrease rapidly and converge to zero for $\tau \rightarrow 0.5$.

These effects are intuitive in the context of the above model. Taxation has two main effects on individual investment into human capital. First, it determines the individually optimal level of human capital via a substitution and an income effect. A higher income tax reduces the rate of return to human capital and thus renders instantaneous consumption more attractive relative to investment. In addition, there is an income effect: Human capital is the only asset from which income can be derived. Therefore less returns to human capital imply that more human capital has to be accumulated to finance consumption in the long-run. Up to intermediate tax levels, the income effect prevails: Taxation increases the individually optimal stock of human capital. But if taxation reduces the yield of human capital so much that its rate of return approaches the discount rate, a cake-eating strategy becomes increasingly attractive, so that further tax increases reduce the optimal level of human capital. Together these points explain the response of the average human capital stock to taxation.

Second, taxation also determines the individually rational rate of adjustment to the optimal stock. The higher the tax is, the smaller are the effects of the stochastic shocks on individual consumption possibilities. Given the risk aversion implicit in our specification of utility, individuals will thus choose a smoother consumption path and a more erratic human capital trajectory, the higher the tax rate is. So taxation increases the variance of the individual human capital stock and (given the independence of individual shocks) consequently the variance of the human capital distribution. Furthermore, the variance of the shocks increases with the current stock size, so that this variance effect is stronger for individuals with a high level of human capital than for those with a low level. Therefore taxation also increases the skewness of the human capital distribution.

So taxation has important and intuitive effects on the distribution of human capital and even in this simple quadratic example, these effects are not constrained to an influence on average human capital.

With the distributional information given by (35), we can analyze the optimal policy. The policy is subject to a budget constraint, that is, the expenditure for the public good have to be financed by the tax revenue, which can be written as

$Z = \int_0^\infty \tau p w f^{eq}(w,\tau) dw$. With (35), this evaluates to

$$Z = \frac{p\tau(p(1-\tau)-\varrho)}{2\alpha(p(1-\tau)+\sigma^2-\varrho)(p(1-\tau)+\sigma^2)}. \tag{36}$$

Calculating social welfare from (34), (35), and (36) yields

$$W(\tau) = \frac{p(p(1-\tau)-\varrho)((2\beta-1)\tau+1)}{4\alpha\varrho\left(p(\tau-1)+\sigma^2-\varrho\right)\left(\sigma^2+p(1-\tau)\right)}. \tag{37}$$

The candidates for an inner optimum of the tax are thus given by[14]

$$\begin{aligned}\tau^* =& \frac{(2\beta-1)\sigma^4+(2\beta-1)(2p-\varrho)\sigma^2+2p\beta(p-\varrho)}{2p\left((2\beta-1)\sigma^2+p\beta\right)} \\ &\pm \frac{\sigma\sqrt{(\sigma^2+\varrho)\left((2\beta-1)\sigma^2+2p\beta\right)\left(-\sigma^2+\varrho+2\beta\left(\sigma^2+p-\varrho\right)\right)}}{2p\left((2\beta-1)\sigma^2+p\beta\right)}.\end{aligned} \tag{38}$$

In the following section, we compare this policy to the recommendations of the representative individual framework and thereby show that the distributional effects can have relevant policy implications.

3.2 Comparison with the Representative Individual

To compare the results of our approach with those gained from a representative individual framework, we derive the optimal steady-state stock of human capital $\hat{w}(\tau)$ from (30) in dependency on the tax and calculate the quantity of the public good from the budget constraint $Z = \tau p \hat{w}(\tau)$. The thus derived value of $\hat{w}(\tau)$ equals the average human capital stock calculated from (35), so that the quantity of the public good in the representative individual framework is identical to (36).

This approach yields the following measure of social welfare.

$$\begin{aligned}\hat{W}(\tau) =& (p(1-\tau)-\varrho)\left(\frac{(\tau-1)^2((2\beta-1)\tau+1)p^3+\left(\varrho-2\sigma^2\right)(\tau-1)((2\beta-1)\tau+1)p^2}{4\alpha\varrho\left(-\sigma^2+\varrho+p(\tau-1)\right)^2\left(\sigma^2+p-p\tau\right)^2}\right. \\ &+\left.\frac{\sigma^2\left(\sigma^2+\varrho-\left((1-2\beta)\sigma^2+2\beta\varrho+\varrho\right)\tau\right)p+\varrho\sigma^2\left(\sigma^2-\varrho\right)}{4\alpha\varrho\left(-\sigma^2+\varrho+p(\tau-1)\right)^2\left(\sigma^2+p-p\tau\right)^2}\right)\end{aligned} \tag{39}$$

Interestingly, this expression is somewhat more complex than (37). Especially, the condition for an inner optimum of τ is a fourth order polynomial in τ, so that it does not provide useful analytic solutions.

[14] Since we have assumed $p(1-\tau) > \varrho$ in the derivation of $f^{eq}(w,z)$, the range of feasible values for τ is $[0,(p-\varrho)/p[$.

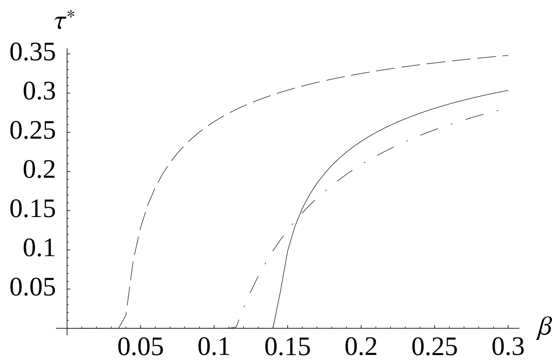

Figure 2: The optimal income tax τ^* in dependency on the marginal utility of the public good β calculated from our approach (solid line) based on (40), the representative individual framework (dashed line), and from (37) (dotted and dashed line).

But comparing (39) and (37) shows that the difference between these expressions depends on τ, so that both approaches will yield different policy recommendations. Whether these differences are substantial enough to necessitate the application of our approach can only be checked numerically for specific examples.

For these examples, it is important to keep in mind that we have used a quadratic approximation of the utility function. As long as the average human capital stock implies a positive consumption with a non-negative marginal utility, which is the case for the above parameterization of the model, this is unproblematic for the representative individual model.

But in our distributional approach, the stochastic shocks will always assure that there are some individuals with very high or very low levels of human capital, which implies a negative marginal utility and a negative consumption, respectively. The latter problem is irrelevant in our setup; the maximal probability that an individual has a negative consumption level is about 0.02%, which has a negligible influence on the policy evaluation.

To get rid of the former problem, we use the following replacement of (29).

$$V(w_0, \tau, Z) = \max_{c \in \mathbb{R}} \int_0^\infty e^{-\varrho t} \max_{\tilde{c} \leq c} \left[\tilde{c} - \alpha \tilde{c}^2 + \beta Z \right] dt. \tag{40}$$

This specification assures that the marginal utility of consumption is always non-negative.

In Figure 2, we have compared the optimal tax calculated from the representative individual framework and from our approach with the parameter values used for calculating Figure 1 for a range of marginal utilities of the public good (β). As Figure 2 shows, the optimal tax is a monotonous function of the marginal utility of the public good in both evaluation approaches. But the representative individual framework prescribes higher taxes for all values of β. This difference can be substantial. For example, with $\beta = 0.15$, the representative individual framework recommends $\tau \approx 0.3$, whereas our approach yields $\tau \approx 0.1$ as the optimal tax rate.

In comparison to this difference, the deviation between the explicitly derived policy recommendation (37) and the approach involving the more sophisticated utility specification (40) is small. Clearly, the problem of not excluding a decreasing marginal utility of consumption has an influence. But this influence is not substantial enough to explain the difference between our concept and the representative individual framework.

The reason why the representative individual framework and our concept result in substantially deviating policy recommendations can be seen from Figure 3, which shows the distribution of human capital for both tax rates. As this figure indicates, the higher tax of the representative individual framework leads to a distribution that has a higher average human capital stock but that is also more skewed. The representative individual framework considers only the higher average value. In contrast to the our approach, it does not take into account that the increase in average human capital is due to a higher prevalence of individuals with very high levels of human capital at the cost of a reduced number of individuals with a medium stock of human capital and not due to a rightward shift of the complete distribution. Since the decreasing marginal utility of the private good implies that the utility gain of increasing human capital is smaller for individuals with a high than for individuals with a medium or low level of human capital, the representative individual framework overstates the benefits of taxation. Consequently, it leads to a higher tax level.

This example shows that it is not necessary to consider highly specialized problems to find substantial differences in the policy recommendations of our approach and of the representative individual concept. These differences are likely to be more pronounced in more complex models. For example, we have not considered the consequences of the policy induced distributional changes on aggregate demand and thus on relative prices. Including such consequences would open additional pathways along which distributional effects could influence social welfare and thus would most likely result in further differences between the implications of our approach and those of the representative individual framework.

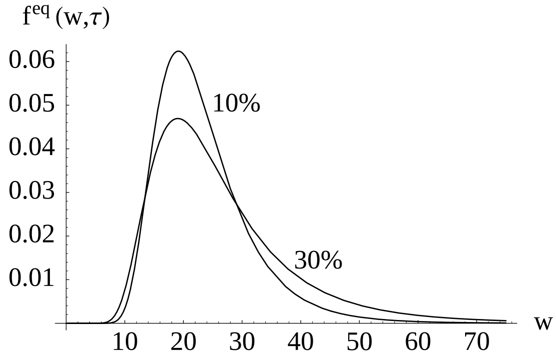

Figure 3: The density of the distribution of human capital for the optimal tax rates calculated from our approach ($\tau = 10\%$) and from the representative individual framework ($\tau = 30\%$).

3.3 Adding Exogenous Heterogeneity

As a final part of this example, we will briefly show how our approach can be extended to cover additional sources of heterogeneity.

For this, we assume that the individual differ not only w.r.t. their human capital but also w.r.t. their discount rates. This difference in intertemporal preferences cannot be explained by individual behavior and is thus not the kind of heterogeneity considered in our analysis. But if we presume that we know the distribution of intertemporal preferences in the economy, we can use our approach to calculate the implied distribution of human capital to characterize the optimal policy.

Assume that the individual discount rates ϱ are normally distributed[15] with $\mathcal{E}(\varrho) = 0.1$, which is consistent with our parameter choice in the preceding sections, and $\mathrm{Var}(\varrho) = 0.025$.

In Figure 4, we have depicted the joint density of human capital w and individual discount rates ϱ that results from (35) for the parameter values used in the preceding sections for two tax levels. As this figure shows, taxation has an additional effect in this framework. It influences the correlation between human capital stocks and intertemporal preferences. The higher the tax is, the more do differences between discount rates induce differences in expected human capital stocks.

Again, this effect is intuitive: As we have discussed above, individuals will respond to a marginal tax increase at a low tax level with higher investments into human capital, and at a high tax level by adjusting their human capital downward.

[15]Of course, this assumption entails that there are individuals with $\varrho < 0$ and $\varrho > p(1-\tau)$. But for the parameter choices considered here, the fraction of such individuals is negligible.

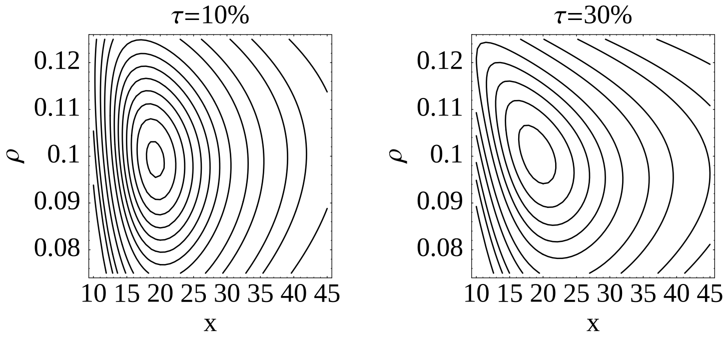

Figure 4: Contour plot of the joint density of the distribution of human capital w and individual discount rates ϱ for two tax levels.

Where the border between "low" and "high" tax level lies, depends on the discount rate of the individual. If there is heterogeneity w.r.t. the discount rate, then the individuals will differ w.r.t. the direction of their investment response to taxation. Thus taxation has an influence on the correlation between ϱ and w.

In Figure 5, we have depicted the optimal policy derived from the different frameworks.[16] As this figure indicates, the additional effect of taxation induces a slightly larger difference between the policy recommendations of our concept and of the representative individual framework. Also the analytically derived optimal tax (37) becomes somewhat less reliable, because the fraction of individuals with decreasing marginal utility of consumption now depends more strongly on taxation. But mostly, the same conclusion holds: The distributional effects of the policy can matter sufficiently to render the representative individual framework questionable for assessing the long-run implications of public policy.

4 Discussion and Conclusions

In this paper we have advanced a concept for evaluating the long-run consequences of public policy in a heterogenous economy that avoids some of the pitfalls of the representative individual framework pointed out in Kirman (1992). This concept is based on endogenously calculating the distribution of individual characteristics, like income or human capital, from a dynamic model of individual behavior. With this distributional information, the welfare effects of public policy can be consistently characterized.

[16]Since the proportion of individuals with $\varrho > p(1-\tau)$ increases with the tax and thus with β, we have consider a somewhat smaller range of values for β here than in the preceding section.

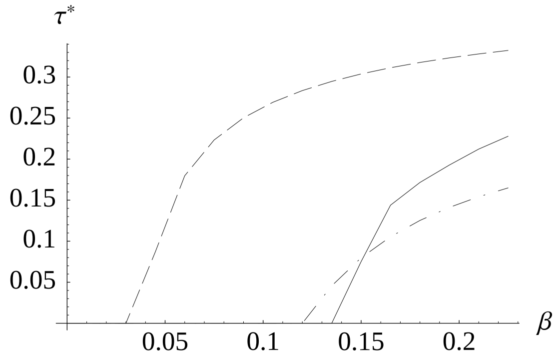

Figure 5: The optimal income tax τ^* for the case of heterogeneity w.r.t. the discount rate in dependency on the marginal utility of the public good β calculated from our approach (solid line) based on (40), the representative individual framework (dashed line), and from (37) (dotted and dashed line).

This approach retains much of the virtues of the representative individual framework. But in contrast to the representative individual concept, it is not troubled by consistency or existence problems, because it is based on an explicit aggregation from the microeconomic to the macroeconomic level.

Furthermore, it captures effects of public policy that are ignored by the representative individual framework, but that are relevant in a long-run setting. In the long-run, a policy does not only influence the welfare of a representative individual, it also determines "who" the representative individual is; it changes the long-run distribution of individual characteristics and thus the characteristics of the "representative" individual. This deviation from the representative individual framework is due to heterogeneous individual responses to public policy and thus only relevant in a long-run perspective. But as our example has shown, it can have considerable implications for policy evaluation.

This influence of public policy on the distribution of individual characteristics is also not accounted for in most multi-agent models. This is the reason why some studies, like Hylland and Zeckhauser (1979) or Christiansen (1981), have concluded that for some aspects of policy evaluation, the representative individual concept provides at least a reasonable approximation to multi-agent models. Our results show that in a long-run perspective, such a vindication of the representative individual framework is not feasible. These studies do not take into account policy induced distributional changes. Thus they neglect a point that is potentially important for assessing a policy's long-run consequences and that cannot be captured in the representative individual framework.

Of course, our approach is subject to several important constraints. The most obvious limitation is that we consider only a specific form of heterogeneity in which the distinctive characteristics of individuals are influenced by their choices. It is clear that there are many cases, like preference heterogeneity, that are not covered by our concept. But still there seem to be many interesting applications, like income or wealth heterogeneity or differences in abilities arising from disparate educational efforts. Furthermore, our concept is still useful if there is both exogenously given and endogenously calculable heterogeneity, for example, if different preferences give rise to different educational choices or if different talents result in different income and wealth. As we have shown in the example, our concept allows to calculate the distribution of the endogenous characteristics if a distribution for the exogenous ones is assumed. Thus data requirements are reduced. More importantly, our policy analysis remains valid, because the exogenous characteristics are not subject to individual decisions and are thus not influenced by public policy. So our approach might be at least of some use in more general cases of heterogeneity.

Another important constraint is that we can only handle problems for which a feedback solution of the individual planning problem can be derived. This problem can be avoided by using intertemporal duality concepts that allow to bypass the necessity of explicitely solving the individual optimization problem, as we have implicitly done in Section 2.3. For stochastic models, such duality concepts can be found in Cooper *et al.* (1995) and in Krysiak (2006). They use the value function as the original representation of preferences and directly derive optimal feedback rules from it by an intertemporal analogue to Roy's Identity or Hotelling's Lemma. Given that our approach requires only the value function and the optimal behavior in feedback form, it is ideally suited to be used in conjunction with these duality concepts. In this way, our concept can be applied to a broad range of economically interesting settings.

Altogether, our approach provides a complement to the representative individual framework. It is especially suited for the analysis of the long-run effects of public policies with a potential influence on the distribution of important individual characteristics, like income or human capital. It retains important similarities with the representative individual framework but differs sufficiently to eliminate the existence problems that plague this framework. Finally, it allows for some additional insights into the effects of public policy that can be relevant for evaluating the policy's long-run consequences.

A Derivation of the Fokker-Planck Equation

In this appendix, we derive the Fokker-Planck Equation (8) from our model of individual behavior (1)–(3) and (7).[17] For this, we track the expected movement of individuals within the state-space of the model, that is, within the set $\mathcal{W}$.

Let ξ be an arbitrary, twice differentiable function of w that vanishes on the boundary of $\mathcal{W}$ (or goes to zero for infinite values of w, if $\mathcal{W}$ is unbounded). Consider an individual that, at time t_0, has the stock w_0. By Itô's Lemma, we can calculate the expected change with time of ξ from (7) as

$$\mathcal{E}\left(\frac{d\xi}{dt}\right) = \mathcal{E}\left(g(w,z)\frac{\partial \xi}{\partial w} + \frac{\sigma^2(w,z)}{2}\frac{\partial^2 \xi}{\partial w^2}\right). \tag{41}$$

Now let $\tilde{f}(w,z,t,w_0)$ be the density of the probability distribution of $w(t)$.[18] Given this density, we can calculate the expected change with time of ξ also as

$$\mathcal{E}\left(\frac{d\xi}{dt}\right) = \int_{\mathcal{W}} \xi \frac{\partial \tilde{f}(w,z,t,w_0)}{\partial t} dw. \tag{42}$$

Using the definition of the expectation operator and comparing (41) and (42) yields

$$\int_{\mathcal{W}} \xi \frac{\partial \tilde{f}(w,z,t,w_0)}{\partial t} dw = \int_{\mathcal{W}} \left(g(w,z)\frac{\partial \xi}{\partial w} + \frac{\sigma^2(w,z)}{2}\frac{\partial^2 \xi}{\partial w^2}\right) \tilde{f}(w,z,t,w_0) dw. \tag{43}$$

By using partial integration, we can express this as[19]

$$\int_{\mathcal{W}} \xi \left(\frac{\partial \tilde{f}(w,z,t,w_0)}{\partial t} + \frac{\partial \tilde{f}(w,z,t,w_0) g(w,z)}{\partial w} - \frac{1}{2}\frac{\partial^2 \sigma^2(w,z)\tilde{f}(w,z,t,w_0)}{\partial w^2}\right) dw = 0. \tag{44}$$

Since the function ξ has been arbitrary, we get

$$\frac{\partial \tilde{f}(w,z,t,w_0)}{\partial t} = -\frac{\partial g(w,z)\tilde{f}(w,z,t,w_0)}{\partial w} + \frac{1}{2}\frac{\partial^2 \sigma^2(w,z)\tilde{f}(w,z,t,w_0)}{\partial w^2}. \tag{45}$$

[17] There are numerous ways to derive a Fokker-Planck equation for a stochastic process. Due to its simplicity and elegance, we use the approach given in Schulten and Kosztin (2000). A more detailed derivation can be found in Toda *et al.* (1991).

[18] For $t = t_0$, this will be a Dirac function. But with $\sigma > 0$, it will be a non-degenerated distribution for all $t > t_0$.

[19] By construction, ξ vanishes on the boundary of $\mathcal{W}$, so that the other terms appearing in the partial integration are zero.

This equation describes the evolution of the density of the probability distribution of the stock of a single individual.

Now let $f_0(w)$ be the relative number of individuals with stock w_0 at time t_0. For each individual, equation (45) yields the density of the individual stocks at time $t > t_0$. From this information, we can calculate the probability that a randomly selected individual has a stock smaller than or equal to w at time t. Let $f(w, z, t)$ be the density of this probability distribution over w. By construction, we have

$$f(w, z, t) = \int_{\mathcal{W}} \tilde{f}(w, z, t, w_0) f_0(w_0) dw_0, \tag{46}$$

and consequently

$$\frac{\partial f(w, z, t)}{\partial t} = \int_{\mathcal{W}} \frac{\partial \tilde{f}(w, z, t, w_0)}{\partial t} f_0(w_0) dw_0. \tag{47}$$

Replacing $\partial \tilde{f}(w, z, t, w_0)/\partial t$ according to (45), leads to

$$\frac{\partial f(w, z, t)}{\partial t} = \int_{\mathcal{W}} \left(-\frac{\partial g(w, z) \tilde{f}(w, z, t, w_0)}{\partial w} + \frac{1}{2} \frac{\partial^2 \sigma^2(w, z) \tilde{f}(w, z, t, w_0)}{\partial w^2} \right) f_0(w_0) dw_0. \tag{48}$$

By exchanging the integral and the differential operators and using the definition (46), we get

$$\frac{\partial f(w, z, t)}{\partial t} = -\frac{\partial g(w, z) f(w, z, t)}{\partial w} + \frac{1}{2} \frac{\partial^2 \sigma^2(w, z) f(w, z, t)}{\partial w^2}. \tag{49}$$

This is the sought Fokker-Planck equation (8).

References

Aoki, M. (1996), *New Approaches to Macroeconomic Modeling* (Cambridge–London–New York: Cambridge University Press).

Aoki, M. (2002), *Modeling Aggregate Behavior and Fluctuations in Economics: Stochastic Views of Interacting Agents* (Cambridge, UK: Cambridge University Press).

Caselli, F. and Ventura, J. (2000), “A Representative Consumer Theory of Distribution”, *American Economic Review*, **90**, 909–926.

Chipman, J. S. and Moore, J. C. (1979), "On Social Welfare Functions and the Aggregation of Preferences", *Journal of Economic Theory*, **21**, 111–1139.

Christiansen, V. (1981), "Evaluation of Public Projects under Optimal Taxation", *Review of Economic Studies*, **48**, 447–457.

Cooper, R. J., Madan, D. B. and McLaren, K. R. (1995), "Approaches to the Solution of Stochastic Intertemporal Consumption Models", *Australian Economic Papers*, **34**, 86–103.

Dorofeenko, V. and Shorish, J. (2005), "Partial Differential Equation Modelling for Stochastic Fixed Strategy Distributed Systems", *Journal of Economic Dynamics and Control*, **29**, 335–367.

Dow, J. and Werlang, S. (1988), "The Consistency of Welfare Judgments with a Representative Consumer", *Journal of Economic Theory*, **44**, 265–280.

Geweke, J. (1985), "Macroeconomic Modeling and the Theory of the Representative Agent", *American Economic Review*, **75**, 206–210.

Hellwig, M. F. (2005), "A Utilitarian Approach to the Provision and Pricing of Excludable Public Goods", *Journal of Public Economics*, **89**, 1981–2003.

Hylland, A. and Zeckhauser, R. J. (1979), "Distributional Objectives Should Affect Taxes but not Program Choice or Design", *Scandinavian Journal of Economics*, **81**, 264–284.

Jerison, M. (1984), "Social Welfare and the Unrepresentative Representative Consumer", Discussion paper, State University of New York at Albany.

Jerison, M. (1994), "Optimal Income Distribution Rules and Representative Consumers", *Review of Economic Studies*, **61**, 739–771.

Kaplow, L. (1996), "The Optimal Supply of Public Goods and the Distortionary Cost of Taxation", *National Tax Journal*, **49**, 513–533.

Kirman, A. P. (1992), "Whom or What Does the Representative Individual Represent?", *Journal of Economic Perspectives*, **6**, 117–136.

Krusell, P. and Ríos-Rull, J.-V. (1999), "On the Size of U.S. Government: Political Economy in the Neoclassical Growth Model", *American Economic Review*, **89**, 1156–1181.

Krusell, P. and Smith, A. A. (1998), "Income and Wealth Heterogeneity in the Macroeconomy", *Journal of Political Economy*, **106**, 867–896.

Krysiak, F. C. (2006), "Stochastic Intertemporal Duality: An Application to Investment under Uncertainty", *Journal of Economic Dynamics and Control*, **30**, 1363–1387.

Kupiec, P. H. and Sharpe, S. A. (1991), "Animal Spirits, Margin Requirements, and Stock Price Volatility", *Journal of Finance*, **46**, 717–731.

Lau, L. J. (1982), "A Note on the Fundamental Theorem of Exact Aggregation", *Economics Letters*, **9**, 119–126.

Mas-Colell, A., Whinston, M. D. and Green, J. R. (1995), *Microeconomic Theory* (New York–Oxford: Oxford University Press).

Merton, R. C. (1971), "Optimum Consumption and Portfolio Rules in a Continuous Time Model", *Journal of Economic Theory*, **3**, 373–413.

Muellbauer, J. (1975), "Aggregation, Income Distribution, and Consumer Demand", *Review of Economic Studies*, **42**, 525–543.

Norman, P. (2004), "Efficient Mechanisms for Public Goods with Use Exclusion", *Review of Economic Studies*, **71**, 1163–1188.

Sandmo, A. (1998), "Redistribution and the Marginal Cost of Public Funds", *Journal of Public Economics*, **70**, 365–382.

Schulten, K. and Kosztin, I. (2000), "Lectures in Theoretical Biophysics", unpublished manuscript, Department of Physics, University of Illinois.

Toda, M., Kubo, R. and Saitô, N. (1991), *Statistical Physics II* (Springer), 2nd edition.

Wright, S. (1945), "The Differential Equation of the Distribution of Gene Frequencies", *Proceedings of the National Academy of Sciences*, **31**, 382–389.