Search, Unemployment, and Age*

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Abstract

We examine a search and matching model of the labor market with overlapping generations of workers. We show that very old workers with the same productivity as younger workers generally earn less. A market solution with Nash-bargained wages is never optimal as old workers search too long compared to the efficient solution. We also examine the effects of policy measures such as hiring subsidies or unemployment insurance. We show that the quantitative impact on employment is always higher once one acknowledges the fact that workers can only work for a finite period of time. Labor market institutions have a differential impact for different age groups.

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1 Introduction

The literature on search and matching models of the labor market is vast and growing (for recent surveys see Mortensen and Pissarides (1999a) and Rogerson et al. (2005)).[1] The crucial assumption is to depart from Walrasian labor markets, where the equilibrium wage equalizes labor demand and labor supply, and to consider a search and matching process where it takes some time for firms to find a worker and some time for a worker to find a job. In these models several important questions have been addressed. When do search models yield an efficient outcome? What are the effects of different assumptions with respect to wage determination? What are the effects of labor market policies?

For ease of computation these search and matching models generally consider workers with "infinite" lives.[2] While this simplifies the analysis substantially, it may still be interesting to explore a model that takes account of the fact that workers live and work for a finite time only. This will enable us to model employment and wages as functions of worker age, which is impossible in standard models. It also enables us to discuss the impact of labor market institutions such as unemployment insurance on the employment rate for different age groups. This is potentially interesting because, for example, the US employment rate differs strongly from the EU-15 value for very young and old workers (see Figure 1), whereas for workers of intermediate age there is almost no difference. One potential explanation may be that labor market institutions have a differential impact on different age groups. Some institutions, like different educational systems or early retirement schemes, obviously have different consequences for different age groups. However, our model shows that the impact of labor market institutions such as unemployment insurance on employment is also age-dependent as they are likely to have their strongest effect on the employment of very young and very old workers.

[1] Important contributions to this literature include Diamond (1982), Pissarides (1985), Mortenson (1986), Wright (1987), Howitt (1988), Hosios (1990), Ljunggvist and Sargent (1998), Ljunggvist and Sargent (2004), den Haan et al. (2005), Lentz and Tranas (2005), Pries and Rogerson (2005), and Shimer (2005).

[2] Albrecht and Axell (1984) consider a model where workers have a constant probability of dying and thus have a finite life expectancy. Nevertheless, worker age plays no role in their model because all unemployed workers and all employed workers are completely identical, irrespective of their date of birth, and have the same life expectancy. Fujimoto (2005) examines a search model with overlapping generations. However, in his model wage offers are exogenous, and unemployed workers only decide whether or not to accept a given wage offer.

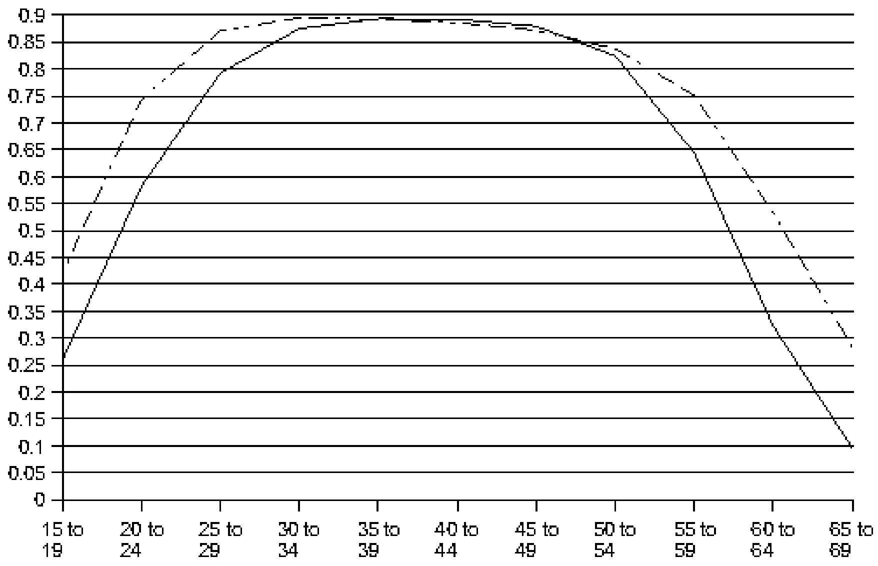

Figure 1: The fraction of employed males for the United States (dotted line) and EU-15 (solid line) as a function of age, average values for the period 1994 to 2004. Source: OECD.

We extend a standard textbook search and matching model found in Romer (2005) along two lines. First, we take into account the fact that the lives of workers are finite and that workers retire at a certain age that is exogenously given in many cases.[3] In particular, we examine a search and matching model with overlapping generations of workers and continuous time. Second, we introduce costs when a new match is formed, which is a frequently used assumption in the literature (compare Mortensen and Pissarides (1999a)). This is based on the realistic observation that it may not be worthwhile to employ workers who are close to retirement, as vocational adjustment may be too costly.

We show that wages remain approximately constant for a specified level of productivity as younger workers grow older. For workers close to retirement, wages are lower than for young workers. Qualitatively, our findings about employment as a function of age correspond to the pattern of employment and age found in the data, e.g. for the United States and EU-15.

It is important to stress that we deliberately consider only one effect of age, namely the effect that workers approach retirement and thus a definite end of the employer-

[3]Many papers consider geometrically distributed life spans (see among others Ljunggvist and Sargent (1998). While in these papers there is a finite expected life span, the expected life span is identical for all workers irrespective of their age and workers can live forever, in principle. Thus age plays no role in these models.

employee relationship as they get older. Obviously, there are other effects that may play a role in reality. For example, experience grows with tenure, which is positively correlated with age. Thus workers will become more productive over time. This may result in wage increases as workers grow older, which do not show up in our model. Moreover, wages may increase with tenure, because firms try to reduce the rate at which workers quit.[4] However, our finding that matches are less valuable if workers are close to retirement, which may lead to lower wages for a given level of productivity, is robust to the incorporation of a productivity level that increases with tenure. Moreover, also in contract-posting models like Burdett and Coles (2003) and Stevens (2004), firms are likely to offer contracts with lower wages for workers closer to retirement.

We also examine the social efficiency of the outcome in our model, since it is unclear whether the results on efficiency derived in the literature carry over to a model where the workers' age is explicitly taken into account. We show that the market equilibrium will never be socially efficient. In particular, the age where workers leave the labor market is inefficiently high. Thus our model can in principle be used to justify regulations such as unemployment insurance because they may enhance the efficiency of the outcome.

The paper is organized as follows. In the next section, we present our model. The steady-state solution to our model is derived in Section 3. In the subsequent section, we examine the socially efficient solution. We discuss the effects of various policy measures in Section 5. To illustrate our findings, we discuss a numerical example in Section 6. Section 7 concludes.

2 Model

We examine a search model with overlapping generations of workers where each generation comprises a large number of individuals. Moreover, there is a large number of infinitely-lived firms that employ workers. In this section, we start with the equations governing the behavior of workers. Then we discuss the equations describing the behavior of firms and the matching process that determines how often workers and firms meet. Finally, the details of the wage-bargaining process are discussed.

[4] See Burdett and Coles (2003) and Stevens (2004).

2.1 Workers

In contrast to the existing literature, we consider workers who can only work for a given period of time. Thus we must always keep track of the age of a worker, which we denote by the variable τ. In general, the wage and the value functions will depend on the age of the worker. Moreover, we introduce an exogenous retirement age T. One could interpret the fact that all workers retire at T as a consequence of a drop in productivity due to age. This drop in productivity could make matches worthless for $\tau > T$. Alternatively, one could interpret T as a barrier set by the legislator, thus prohibiting employment or making it unattractive due to high retirement benefits. We assume that the value of retiring at T, i.e. the discounted value of all future utility flows, amounts to $R/r > 0$.

At each point in time, a new generation of workers aged 0 enters the labor market, i.e. they leave school. They are initially unemployed but immediately start to search for a job. Consequently, τ can be best thought of as the number of years after education has been completed. Nevertheless we will always refer to τ as the age. We assume that each generation comprises the same number of workers. Thus, the size of the population of workers is constant over time. It will be convenient to normalize the number of workers in each generation to 1. Consequently, the overall size of the population of workers is T.

Workers maximize their discounted sum of instantaneous utility, where the discount rate is $r > 0$. Instantaneous utility is identical to wage $w(\tau)$ if the worker is either employed or equals B ($B > 0$). We consider two interpretations of the utility B. First, it can be interpreted as the output of home production or as the utility derived from leisure. Second, it could also comprise transfers from the state. This distinction is crucial for our analysis of welfare, which we undertake in Section 4.

Each existing match creates the same flow of output A, i.e. for each period dt the output $A\,dt$ is produced. Whenever a firm hires a new worker, it incurs some instantaneous training costs $k > 0$ when the worker starts working.[5] These costs may also be explained by the lower productivity of workers who have been hired only very recently.

Before agreeing on a match, a worker and a firm can freely distribute these costs among themselves. We introduce $\sigma > 0$, which represents the transfer payment (or

[5] These costs correspond to the creation costs C in Mortensen and Pissarides (1999a) or Mortensen and Pissarides (1999b).

wage discount) that the worker pays to the firm.[6] Note that, e.g., in Mortensen and Pissarides (1999a), the training costs are shared between the firm and the worker through a two-tier wage structure. The worker agrees on an initial wage discount, which is used to cover part of the training costs. In our model the transfer payment can also be given the interpretation of a wage discount. It would be straighforward to formulate our model in terms of a two-tier wage structure.

The costs $k > 0$ reflect the realistic feature that firms do not hire a worker who is very close to retirement age T. Thus we introduce a second date $\widehat{T}$, which will be determined endogenously. When an unemployed worker reaches the age of $\widehat{T}$, she will never find a match that creates a positive value because the costs k would be too high compared to the output created over the remaining period up to retirement. Thus $\widehat{T}$ is the age at which the net surplus of a match, i.e. the sum of the values of a match for the firm and the worker reduced by the costs k, is zero. We assume that those unemployed workers older than $\widehat{T}$ leave the labor market and do not partake in search activities. One reason may be that there are very small costs for searching, which we do not model explicitly. These costs may induce workers that will never find a job to desist from any search activity.

We also make the assumption often used in the literature that there is a constant exogenous hazard rate $b > 0$.[7] Each employed worker loses her job in each period dt with a probability of $b\,dt$. We use a to denote the rate at which workers find jobs, which is identical for all unemployed workers looking for a job. This rate will be determined later by the matching technology.

We now turn to the "asset-price" equations for workers with $\tau \leq \widehat{T}$, which can be derived from the respective Bellman equations[8]

$$\begin{aligned} rV_E^-(\tau) &= w^-(\tau) + b\left(V_U^-(\tau) - V_E^-(\tau)\right) + \dot{V}_E^-(\tau) && (1)\\ rV_U^-(\tau) &= B + a\left(V_E^-(\tau) - V_U^-(\tau) - \sigma\right) + \dot{V}_U^-(\tau), && (2)\end{aligned}$$

where $V_E^-(\tau)$ and $V_U^-(\tau)$ denote the value function for a worker of age τ who is employed

[6] This assumption implies that any worker-firm relationship that would deliver some positive surplus is agreed upon. If the firm carried the costs alone, then it would be possible that a match would create a positive surplus,but cannot be formed.

[7] An endogenous destruction of existing matches due to productivity shocks à la Mortensen and Pissarides (1994) would complicate the analysis considerably, as the equilibrium reservation productivity would depend on the age of the worker.

[8] For example, the first differential equation corresponds to the Bellman equation: $V_E^-(\tau) = w^-(\tau)\,d\tau + \frac{1}{1+r\,d\tau}\left(b\,d\tau\, V_U^-(\tau+d\tau) + (1-b\,d\tau)V_E^-(\tau+d\tau)\right)$ for $d\tau \to 0$.

or unemployed, respectively.[9] The wage paid to workers who are τ years old is denoted by $w^-(\tau)$. The superscript "$-$" is used to indicate that $\tau < \widehat{T}$. We use a dot "$\cdot$" to denote the derivative with respect to τ.

Equation (1) can be interpreted as follows: If someone holds an asset with the value $V_E^-(\tau)$, he incurs opportunity costs $rV_E^-(\tau)$. In equilibrium, these costs must equal the sum of the dividend stream $w^-(\tau)$ and the expected change of the asset's value. The asset's value may change by $V_U^-(\tau) - V_E^-(\tau)$, which happens at a rate b. Moreover, the term $\dot{V}_E^-(\tau)$ captures the change in value as the worker gets older.

According to (2), an unemployed worker's lifetime utility may change because, during each period dt, there is a probability $a\,dt$ of her meeting a firm. If the firm and the worker agree to form a match, her lifetime utility changes by $V_E^-(\tau) - V_U^-(\tau)$. However, before starting to work she has to agree to an immediate payment σ, which goes to the firm to cover a share of its training costs k.

For $\tau > \widehat{T}$, workers who lose their jobs do not search for a new position. The respective "asset-price" equations are

$$rV_E^+(\tau) = w^+(\tau) + b\left(V_0^+(\tau) - V_E^+(\tau)\right) + \dot{V}_E^+(\tau) \qquad (3)$$
$$rV_0^+(\tau) = B + \dot{V}_0^+(\tau), \qquad (4)$$

where $V_0^+(\tau)$ is the value function for an unemployed worker who is not searching for a job. We use the subscript 0, as opposed to U, to indicate that the respective worker does not seek employment. The superscript "+" is introduced to make it clear that the period $\tau > \widehat{T}$ is considered. We also have to take into account the boundary conditions $V_0^+(T) = V_E^+(T) = \frac{R}{r}$. In addition, the value functions must be continuous at $\widehat{T}$, which yields $V_E^-(\widehat{T}) = V_E^+(\widehat{T})$ and $V_U^-(\widehat{T}) = V_0^+(\widehat{T})$.

Note that the term R/r, which denotes the stream of retirement benefits each worker receives when she is older than T, plays no role in our model, since workers receive these benefits irrespective of their history of employment, i.e. a worker who has been unemployed all her life will receive the same stream of benefits as a worker who has worked for a long time.

[9] Note that $V_E^-(\tau)$ represents the value function for a worker who has already paid the transfers σ. Therefore the value function for a worker aged τ who agrees to form a new match is given by $V_E^-(\tau) - \sigma$.

2.2 Firms

Firms maximize their discounted sum of profits, applying the same discount rate $r > 0$ as workers. From each match, the difference of the output produced by this match and the wage paid to the respective worker accrues to the firm.

We assume that each match generates an output flow of A, which we assume to be higher than the flow of unemployment benefits B. This implies that

$$\Delta A := A - B > 0. \tag{5}$$

Firms are able to create vacancies at zero cost. However, there are flow costs c for maintaining a vacancy, i.e. each vacancy involves a recruiting cost of $c\,dt$ for each period dt in which it is maintained. For each vacancy, firms find a worker at a rate $\alpha > 0$. This variable will be determined endogenously by the matching technology.

We have already mentioned that each newly formed match creates some training costs k, which accrue to the firm immediately after the match has been formed and before the worker starts working. These costs may represent transaction costs from forming a new match, or they may stem from vocational adjustment, i.e. the worker is not very productive at the beginning or a new match requires some costly training. However, to make the analysis as simple as possible we assume that these costs accrue instantaneously.

Let us now turn to the first "asset-price" equation for a firm

$$rV_F(\tau) = A - w(\tau) + b\,(V_V - V_F(\tau)) + \dot{V}_F(\tau). \tag{6}$$

The value of a filled vacancy for a firm is given by $V_F(\tau)$, whereas the value of a vacancy is given by V_V. Note that the value of a vacancy does not depend on τ, as the age of the worker who will fill the vacancy is unknown.

The interpretation of this differential equation is straightforward. If one holds a unit of an asset worth $V_F(\tau)$, opportunity costs of $rV_F(\tau)$ arise. In equilibrium these must equal the sum of the dividend stream and the expected change in value. The dividend stream is given by the output from a match A minus the wage the firm has to pay. Additionally, the asset may change its value by $V_V - V_F(\tau)$ because the match is split. This occurs at a rate b. If no shock occurs, the asset changes its value because the worker is getting older. This is captured by the term $\dot{V}_F(\tau)$. As (6) describes the value of a filled position for a firm after the training costs k have accrued, it does not depend on k.

We must also take into account the boundary condition $V_F(T) = 0$, which states that a firm attaches no value to a match at $\tau = T$ because the match is dissolved with certainty.

The following "asset-price" equation describes the value of a vacancy for a firm

$$rV_V = -c + \alpha \left(\frac{1}{u(\widehat{T})} \int_0^{\widehat{T}} V_F^-(\tau)\mu_U(\tau)\, d\tau + \sigma - k - V_V \right), \tag{7}$$

where $\mu_U(\tau)$ is the fraction of unemployed workers at age τ. Overall unemployment[10] is then given by

$$u(\widehat{T}) := \int_0^{\widehat{T}} \mu_U(\tau)\, d\tau. \tag{8}$$

Henceforth we will often omit the argument $\widehat{T}$ and thus write u instead of $u(\widehat{T})$.

Equation (7) has a straightforward interpretation as $\frac{1}{u} \int_0^{\widehat{T}} V_F^-(\tau)\mu_U(\tau)\, d\tau + \sigma - k$ denotes the expected value of a new match for a firm and the firm finds new workers at rate α. Because vacancies can be created without cost, the value of a vacancy must equal zero in equilibrium ($V_V = 0$), otherwise it would be profitable to either reduce the amount of vacancies or to increase it.

2.3 Matching Technology

Let us now turn to the matching technology. As is common in the literature, we assume that matches are created according to an exogenously given matching function $m(u, v)$ with constant returns to scale, where v denotes the number of vacancies provided by firms and u is the number of unemployed searching for a job. The expression $m(u, v)\, dt$ gives the number of worker-firm matches that occur in every infinitesimally small period of time dt. It is then convenient to introduce the labor market tightness $\theta := v/u$. We also employ the frequently used assumptions that $m_u(.) > 0$, $m_v(.) > 0$, $m_{uu}(.) < 0$ and $m_{vv}(.) < 0$. In addition, we postulate $\lim_{u\to\infty} m(u, v) = \infty$ and $\lim_{v\to\infty} m(u, v) = \infty$. One example for a function satisfying these assumptions is a Cobb-Douglas matching function.

[10]Note that the term "overall unemployment" corresponds to the number of workers who have no job but are searching for one. Consequently, it does not include the workers who have no job and are older than $\widehat{T}$.

As the number of workers finding a job must always equal the number of firms hiring a worker, the following equation holds

$$au = m(u, v) = \alpha v. \tag{9}$$

Since $m(u, v)$ has constant returns to scale, a and α can be written as functions of θ with $a'(\theta) > 0$, $\lim_{\theta\to\infty} a(\theta) = \infty$, $a(0) = 0$, $\alpha'(\theta) < 0$, $\lim_{\theta\to 0} \alpha(\theta) = \infty$ and $\lim_{\theta\to\infty} \alpha(\theta) = 0$.

2.4 Wage Bargaining

Since in matching models a successful match creates rents for firms and workers, the wage that distributes these rents between the worker and the firm is not uniquely determined, as, for both parties, a variety of Pareto-efficient outcomes exists. One approach that can be justified by Nash-bargaining is to assume that the wage is chosen in such a way that a fraction γ of the expected present value of all future benefits from a match goes to the worker, whereas $1-\gamma$ of these benefits accrue to the firm. We assume that the bargaining powers of the worker and the firm remain constant over time. At each point in time, the wage is re-negotiated such that the following conditions always hold

$$\begin{aligned} V_E^-(\tau) - V_U^-(\tau) =& \gamma X^-(\tau) \\ V_E^+(\tau) - V_0^+(\tau) =& \gamma X^+(\tau), \end{aligned} \tag{10}$$

where we have introduced the joint surplus of a match for the worker and the firm as

$$X^-(\tau) \quad := \quad V_F^-(\tau) + V_E^-(\tau) - V_U^-(\tau) \tag{11}$$

$$X^+(\tau) \quad := \quad V_F^+(\tau) + V_E^+(\tau) - V_0^+(\tau). \tag{12}$$

An alternative assumption would be that wages are agreed upon when the match is formed and remain constant afterwards. However, this assumption would make the analysis more complex as we would always have to keep track of the hiring date. We will return to this point in our conclusions.

According to the boundary conditions for $V_F^+(\tau)$, $V_E^+(\tau)$ and $V_0^+(\tau)$, the joint surplus $X^+(\tau)$ approaches zero when τ approaches T. This is intuitive, as the match is solved with certainty at T.

Moreover, since the value functions are continuous at $\widehat{T}$, this property translates into the continuity of the joint surplus, i.e. $X^-(\widehat{T}) = X^+(\widehat{T})$. Having specified our model, we can now characterize the steady state.

3 Steady State

In the following, we will attempt to identify a steady state. In a steady state, the number of unemployed u and the number of vacancies v do not change over time. More specifically, we postulate that the fraction of unemployed workers of a certain age τ, $u(\tau)$, must be constant over time. For example, if at a certain point in time 5% of those aged 40 are unemployed, this property must also hold for all other points in time. Similarly, $\widehat{T}$ must also be a constant. In particular, this implies that all value functions of workers do not depend on time t but only on age τ.

3.1 Employment as a Function of Age

In the next subsections we will see that the amount of vacancies created by firms depends on the age distribution among those looking for a job. Thus we analyze how the unemployment rate depends on age τ. We start with the case $\tau < \widehat{T}$:

$$\begin{aligned}\dot{\mu}_E^-(\tau) &= a\mu_U^-(\tau) - b\mu_E^-(\tau)\\ \dot{\mu}_U^-(\tau) &= b\mu_E^-(\tau) - a\mu_U^-(\tau).\end{aligned}$$

The change in the employment rate for a certain age τ is given by the number of those aged τ who are unemployed times the rate a at which those unemployed workers find a job. Additionally, workers who are employed lose their jobs at rate b.

We have already mentioned our assumption that all workers are initially unemployed, which gives us the boundary conditions $\mu_E^-(0) = 0$ and $\mu_U^-(0) = 1$. It can easily be seen that the above system of differential equations has the following solutions:

$$\begin{aligned}\mu_U^-(\tau) &= \frac{b}{a+b} + \frac{a}{a+b}e^{-(a+b)\tau}\\ \mu_E^-(\tau) &= \frac{a}{a+b}(1 - e^{-(a+b)\tau}).\end{aligned}$$

Now we can compute the overall number of unemployed workers for all generations, which is given by

$$u = \int_0^{\widehat{T}} \mu_U(\tau)\, d\tau = \frac{b}{a+b}\widehat{T} + \frac{a}{(a+b)^2}\left(1 - e^{-(a+b)\widehat{T}}\right).$$

This leaves us with the case of $\tau > \widehat{T}$. For $\tau > \widehat{T}$, all workers without employment leave the labor market and do not search for a new job.

$$\dot{\mu}_E^+(\tau) = -b\mu_E^+(\tau) \qquad \mu_U^+(\tau) = 0 \qquad \dot{\mu}_0^+(\tau) = b\mu_E^+(\tau).$$

Recognizing that the employment rate must be continuous at $\widehat{T}$, which can be formally stated as $\mu_E^+(\widehat{T}) = \mu_E^-(\widehat{T})$ or $\mu_0^+(\widehat{T}) = \mu_U^-(\widehat{T})$, we obtain the solutions

$$\begin{aligned} \mu_E^+(\tau) &= \mu_E^-(\widehat{T})e^{-b(\tau-\widehat{T})} \\ \mu_0^+(\tau) &= 1-\mu_E^+(\tau). \end{aligned}$$

To sum up, employment as a function of age is increasing up to the age $\widehat{T}$. Since workers who are older than $\widehat{T}$ cannot find a job and some matches are always separated by the exogenous shocks, employment is declining for $\tau \geq \widehat{T}$.

3.2 Transfers σ

In the following we analyze the size of the transfer σ. Note that the total surplus of a match, just before the worker starts working, is given by $X^-(\tau) - k$.[11] A fraction γ of this surplus goes to the worker, the rest accrues to the firm. If a new match is formed, the worker's lifetime utility increases by $V_E^-(\tau) - V_U^-(\tau) - \sigma$ and the firm's profits increase by $V_F^-(\tau) - V_V + \sigma - k$. If a fraction γ of the overall surplus $X^-(\tau) - k$ is distributed to the worker, the following equation must hold

$$V_E^-(\tau) - V_U^-(\tau) - \sigma = \gamma(X^-(\tau) - k). \tag{13}$$

Since $V_E^-(\tau) - V_U^-(\tau) = \gamma X^-(\tau)$, we obtain

$$\sigma = \gamma k. \tag{14}$$

Hence the higher the workers' bargaining power, the higher the share of the costs σ paid by them. This may seem counterintuitive at first sight, but can be explained as follows. For example, if the workers' bargaining power is $\gamma = 1$, then they will be able to extract the whole surplus from the match $X(\tau)$ later. Thus a firm will only be willing to employ a worker if she fully compensates the firm for the costs k. By contrast, for $\gamma = 0$ a worker is only willing to accept a match if the firm pays for all costs k. Note that our results are in line with Mortensen and Pissarides (1999a) if we take account of the fact that the transfer payments in our model correspond to the difference between the insider wage and the initial wage in their article.

[11] By contrast, the total surplus of a match where the worker has already started working is $X^-(\tau)$.

3.3 The Case $\tau > \widehat{\mathbf{T}}$

Now we consider the case $\tau > \widehat{T}$, i.e. the case where unemployed workers are too old to find a job. It is straightforward to show that the system of the differential equations (3), (4), and (6) can be transformed into a differential equation for the joint surplus

$$rX^{+}(\tau) = \Delta A - bX^{+}(\tau) + \dot{X}^{+}(\tau). \tag{15}$$

Together with the boundary condition $X^{+}(T) = 0$ this differential equation solves to

$$X^{+}(\tau) = \frac{\Delta A}{b+r}\left(1 - e^{-(b+r)(T-\tau)}\right). \tag{16}$$

This function is decreasing in τ. As the worker gets older, the maximum time span the worker will be employed by the firm is decreasing. This makes the match continuously less valuable. When τ approaches T, the match is completely worthless as all matches end at T.

The wage can be obtained by plugging the above solution and $V_F^{+}(\tau) = (1-\gamma)X^{+}(\tau)$ into (6):

$$\begin{aligned} w^{+}(\tau) &= A - (b+r)V_F^{+}(\tau) + \dot{V}_F^{+}(\tau) \\ &= A - (b+r)(1-\gamma)X^{+}(\tau) + (1-\gamma)\dot{X}^{+}(\tau). \end{aligned}$$

Straightforward computations yield

$$w^{+}(\tau) = \gamma A + (1-\gamma)B. \tag{17}$$

If productivity A is higher, then workers earn more. In addition, the wage increases in proportion with a higher utility flow of unemployed workers, B. The wage is also strictly increasing in the worker's bargaining power γ. However, it is interesting to note that here the wage is independent of the worker's age. We will see later that this property does not hold for $\tau < \widehat{T}$.

3.4 The Case $\tau = \widehat{\mathbf{T}}$

Before we start with our analysis of the case $\tau < \widehat{T}$, we consider the case $\tau = \widehat{T}$, which will allow us determine the equilibrium value of $\widehat{T}$. A match will only be formed if the expected surplus $X(\tau)$ is larger than the costs k. At age $\widehat{T}$, both parties are indifferent between starting a relationship and rejecting the match. Formally this can be stated as

$$k = X^+(\widehat{T}). \tag{18}$$

If we combine (16) and (18), we can solve for the equilibrium value of $\widehat{T}$

$$\widehat{T}_{Eq} = \frac{1}{b+r} \ln \left(1 - \frac{(b+r)k}{\Delta A}\right) + T. \tag{19}$$

In the following, we will always assume that this expression has a solution, which can be formally stated as $\Delta A > (b+r)k$, and that this solution is positive. If this were not the case, then the costs k would be so large that it would never be profitable to hire a worker.

It is instructive to discuss how $\widehat{T}_{Eq}$ depends on the parameters of the model. If b or r are increasing, then $\widehat{T}_{Eq}$ is decreasing because it is less valuable to form a match. The same holds for an increase in B or a decrease in A.

3.5 The Case $\tau < \widehat{\mathbf{T}}$

Using the joint surplus $X^-(\tau)$, the system of three differential equations (1), (2) and (6) can be reduced to one differential equation

$$(b + r + a\gamma)X^-(\tau) = \Delta A + a\gamma k + \dot{X}^-(\tau) \tag{20}$$

with the boundary conditions $X^-(\widehat{T}) = X^+(\widehat{T}) = k$.

The differential Equation (20) has the following solution

$$\begin{aligned} X^-(\tau) &= \int_\tau^{\widehat{T}} (\Delta A + a\gamma k) e^{-(b+r+a\gamma)(\widehat{T}-\tau)} \, d\tau + e^{-(b+r+a\gamma)(\widehat{T}-\tau)} X^+(\widehat{T}) \\ &= \frac{\Delta A + a\gamma k}{b+r+a\gamma} \left(1 - e^{-(b+r+a\gamma)(\widehat{T}-\tau)}\right) + e^{-(b+r+a\gamma)(\widehat{T}-\tau)} k. \end{aligned} \tag{21}$$

Since $X^-(\tau) - k$ represents the value of a new match, which is important for the provision of vacancies, we compute

$$X^-(\tau) - k = \frac{\Delta A - (b+r)k}{b+r+a\gamma} \left(1 - e^{-(b+r+a\gamma)(\widehat{T}-\tau)}\right). \tag{22}$$

It is important to note that $X^-(\tau) - k$ is monotonously decreasing in a. If a is large, it is easier for a worker to find a new job, which makes her attribute less value to a given new match. Note that $\frac{\Delta A - (b+r)k}{b+r+a\gamma}$ represents the value of a new match for a hypothetical worker who lives indefinitely. Thus the value of a new match for a worker of age τ can be written as the product of the respective value for an infinitely lived worker and an age-dependent factor.

In the Appendix we derive the wage for workers who are younger than $\widehat{T}$:

$$w^{-}(\tau) = A - (1-\gamma)\left((b+r)\frac{\Delta A + a\gamma k}{b+r+a\gamma} + a\gamma\frac{\Delta A - (b+r)k}{b+r+a\gamma}e^{-(b+r+a\gamma)(\widehat{T}-\tau)}\right). \quad (23)$$

It is instructive to evaluate the wage for the two polar cases $\gamma = 0$ and $\gamma = 1$. For $\gamma = 0$, which represents the case where all bargaining power lies with the firm, the firm is able to extract all of the surplus. Then the wage is chosen in such a way that the worker is paid precisely her reservation wage $w^{-}(\tau) = B$. If $\gamma = 1$, workers have maximum bargaining power and thus receive the whole output $w^{-}(\tau) = A$. For $\tau = \widehat{T}$, we obtain $w^{-}(\widehat{T}) = \gamma A + (1-\gamma)B$. Hence the wage is a continuous function at $\widehat{T}$.

In order to examine how age affects the wage of a worker, we compute

$$\dot{w}^{-}(\tau) = -(1-\gamma)\gamma a\left(\Delta A - (b+r)k\right)e^{-(b+r+a\gamma)(\widehat{T}-\tau)}. \quad (24)$$

As $\Delta A > (b+r)k$, it is easy to see that the derivative of $w^{-}(\tau)$ with respect to age τ is always negative (unless $\gamma = 0$ or $\gamma = 1$). The wages of workers are thus decreasing as they get older. However, wages are approximately constant for young workers (small $(b+r+a\gamma)(\widehat{T}-\tau)$), they decrease substantially only for very old workers. We discuss this point in more detail in Section 7.

3.6 Endogenous Vacancies

Now we analyze a firm's decision to create vacancies in more detail. Since vacancies can be created without cost, optimality requires that the value of a vacancy be zero in equilibrium. By inserting $V_V = 0$ into (7), we obtain

$$c = \alpha \cdot (1-\gamma) \cdot \frac{1}{u}\int_0^{\widehat{T}} \left(X^{-}(\tau) - k\right)\mu_U^{-}(\tau)\,d\tau. \quad (25)$$

This equation has the following interpretation: The costs of maintaining a vacancy amount to c. The rate at which firms find workers is given by α. Moreover, a fraction $1-\gamma$ of the expected value of $X^{-}(\tau) - k$ corresponds to the expected benefits of a new match for a firm. Hence the right-hand side of (25) represents the expected benefits from maintaining a vacancy. In equilibrium the costs and the benefits of maintaining a vacancy must balance.

3.7 Uniqueness and Existence

So far, we have derived the conditions that must be fulfilled in a steady state. In the Appendix we show that these are equivalent to

Proposition 1
The steady state is characterized by a labor market tightness θ and an age $\widehat{T}$ such that

$$\frac{c}{\alpha(\theta)} = (1-\gamma)\cdot(\Delta A-(b+r)k)\cdot\frac{1}{u(\widehat{T})}\int_0^{\widehat{T}} u(\tau)e^{-(b+r+a(\theta)\gamma)(\widehat{T}-\tau)}\,d\tau \qquad (26)$$

$$\widehat{T}_{Eq} = \frac{1}{b+r}\ln\left(1-\frac{(b+r)k}{\Delta A}\right)+T. \qquad (27)$$

However, it is unclear as yet whether such a steady state actually exists and, if so, whether it is unique. This amounts to verifying whether a unique solution for θ exists. In the Appendix, we show

Proposition 2
A steady state always exists.

With regard to uniqueness, it is theoretically possible that more than one steady-state might exist.[12] The left-hand side of Equation (26) is always decreasing in θ. This is plausible because it displays the average cost of waiting for a new match, where it takes $1/\alpha$ on average for a vacancy to be filled. A priori the impact of an increase in θ, and thus in a, is unclear for the right-hand side of Equation (26). While we have already argued that an increase in a reduces the value of a new match for a given age τ, an increase in a also tends to decrease the age of workers filling a vacancy. Since younger workers are more valuable, this effect increases the expected value of a new match, $\frac{1}{u}\int_0^{\widehat{T}}(X^-(\tau)-k)\,\mu_U^-(\tau)\,d\tau$.

Notably for very small values of γ, which means that, for a given age τ, the impact of a on the value of a match is very small, a second effect may predominate, where a higher value of a decreases the age of newly-matched workers, which in turn increases the expected value of a match.[13]

[12] For example, it can be verified numerically that multiple equilibria exist for $m(u,v)=u^{0.1}v^{0.9}$, $r=0.04$, $b=0.1$, $A=2$, $B=0$, $k=3$, $c=3$, $\gamma=0$ and $T=5$.
[13] However, for an infinitely high value of a the expected value of a match always converges to zero, unless $\gamma=0$.

Note that for $T \to \infty$, which represents the standard case of infinitely lived workers, (26) can be written as

$$\frac{c\theta}{a(\theta)} = (1-\gamma)\frac{\Delta A - (b+r)k}{b+r+a(\theta)\gamma}. \tag{28}$$

For $k = 0$, this simplifies to

$$\frac{c\theta}{a(\theta)} = (1-\gamma)\frac{\Delta A}{b+r+a(\theta)\gamma}, \tag{29}$$

which is equivalent to the result described, e.g., in Rogerson et al. (2005). In line with the existing literature, a unique steady state always exists for $T \to \infty$. In the following, we assume that T is sufficiently high so that a unique steady state exists.

4 Efficiency

In the Appendix we derive the socially optimal solution. The social planner chooses labor-market tightness θ and date $\widehat{T}$ where unemployed workers stop searching for a job. The social planner cannot influence the matching process.

The social planner's objective is to maximize the discounted sum of net output, which is given by the sum of output and unemployment benefits B minus the recruiting costs, c, and the training costs, k. Thus we consider the case where unemployment benefits B comprise only the benefits from leisure or the output from home production. In the Appendix we show that

Proposition 3
The socially optimal solution is characterized by

$$c = a'(\theta)\left(\frac{1}{u}\int_0^{\widehat{T}}\left(x^-(\tau)-k\right)\mu_{U,t}(\tau)d\tau\right) \tag{30}$$

$$x^-(\tau)-k = \frac{\Delta A + c\theta - (b+r)k}{a(\theta)+b+r} - \frac{\Delta A - (b+r)\left(\frac{c\theta}{a(\theta)}+k\right)}{a(\theta)+b+r}e^{-(a(\theta)+b+r)(\widehat{T}-\tau)} \tag{31}$$

$$\widehat{T}_{Opt} = \frac{1}{b+r}\ln\left(1-\frac{(b+r)\left(k+\frac{\theta c}{a(\theta)}\right)}{\Delta A}\right)+T. \tag{32}$$

We immediately obtain the following important corollary:

Corollary 1
$\widehat{T}_{Opt} < \widehat{T}_{Eq}$.

Hence the socially optimal plan implies that old workers who are not employed should stop searching earlier than in the market solution. When workers decide on whether to leave the labor market, several externalities are involved. If the workers aged $\widehat{T}$ continue to search for a bit longer, they reduce the likelihood of other workers finding a job. In addition, they increase the likelihood of firms finding a match. These are fairly standard externalities often found in search and matching models. However, there is also an additional negative externality: If workers continue searching for a job, they increase the average age of a worker forming a new match, which decreases the average value of a match for firms. Overall, Corollary 1 implies that the negative externalities dominate and workers search for longer than the social optimum.

With respect to the provision of vacancies, the steady state will in general not be efficient, and no simple condition for efficiency like the Hosios condition can be found.[14] However, it is important to note that for $T \to \infty$ we obtain

$$x^{-}(\tau) - k = \frac{\Delta A + c\theta - (b+r)k}{a(\theta) + b + r}, \tag{33}$$

which implies

$$c = a'(\theta)\frac{\Delta A + c\theta - (b+r)k}{a(\theta) + b + r}. \tag{34}$$

If the Hosios condition holds which stipulates that the employer's bargaining power $1-\gamma$ must be equal to the elasticity of the matching function with respect to vacancies, this equation is equivalent to (28). Thus, for workers with infinite lives we obtain the standard result that the steady state is socially efficient if and only if the Hosios condition holds.

5 Effects of Policy

In this section we describe the qualitative effects of some labor market policy interventions. For this purpose it will be useful to show that the right-hand side of equation (26) is increasing with $\widehat{T}$.

Proposition 4
$\frac{1}{u(\widehat{T})} \int_0^{\widehat{T}} u(\tau) e^{-(b+r+a\gamma)(\widehat{T}-\tau)}\, d\tau$ *is an increasing function of* $\widehat{T}$.

The proof is given in the Appendix. We immediately obtain the following corollary:

[14] This condition was first derived by Hosios (1990) for a class of search models.

Corollary 2

If one takes into account that workers do not live indefinitely, then θ, a, and overall employment are always lower compared to the case of infinitely lived workers ($T \to \infty$) for otherwise identical parameter values.

If workers only live for a finite period of time, this reduces the value of matches for firms. Hence they provide fewer vacancies compared to the case of infinitely lived workers, which makes it less easy to find a job (lower a). Consequently, the fraction of employed workers, $1/T \int_0^T \mu_E(\tau)\, d\tau$, is also lower for workers with finite lives than for workers with infinite lives.

5.1 Unemployment Insurance

In the following, we interpret at least part of the utility flow B as unemployment benefits that can be chosen by the government. An increase in B decreases $\Delta A - (b+r)k$, which is proportional to the value of a new match for the case of a worker with an infinite life. In addition to this standard effect, another effect occurs in our model.

An increase in B also causes a decrease in $\widehat{T}$. This is quite plausible, as higher unemployment insurance makes employment less attractive and thus induces old unemployed workers to stop searching earlier.

According to Proposition 4 an increase in B and the corresponding decrease in $\widehat{T}$ also has a negative impact on the value of a match, $X^{-}(\tau) - k$. This further decreases the average value of a new match. Therefore the impact of an increase in B on the value of a new match is higher once one recognizes that workers do not live forever. Since the expected value of a new match is lower for any value of θ, the equilibrium value of θ, and thus a, is lower. It is important to note that the decrease in a as a result of an increase in unemployment benefits is more substantial for workers with finite lives compared to workers living indefinitely because of the additional effect on $\widehat{T}$ described above.

Now we turn to the effect of an increase in unemployment insurance on employment. An increase in B lowers a, which has a negative impact on employment $\int_0^{\widehat{T}} \mu_E^{-}(\tau)\, d\tau + \int_{\widehat{T}}^{T} \mu_E^{+}(\tau)\, d\tau$. In addition, the decrease in $\widehat{T}$ further decreases employment because workers stop searching earlier. While the first effect is relatively standard, the second effect is only present in a model involving workers with finite lives. It strengthens the negative effect of unemployment insurance on employment. Hence the impact of an increase in unemployment insurance on employment is always larger with finite-life workers than it is with infinite-life workers.

We now turn to the consequences of a rise in unemployment benefits for different age groups. In the Appendix, we show that

Proposition 5
$\frac{d\mu_E^-(\tau)}{dB}$ *has a local minimum at* $\tau = 1/a$ *and at* $\tau = \widehat{T}$.

This implies that an increase in unemployment insurance has its strongest impact on the employment rate of workers who are relatively young ($\tau = 1/a$) and relatively old ($\tau = \widehat{T}$). Intuitively, an increase in B and a corresponding decrease in a will harm very young workers in particular, a large fraction of them being unemployed. Furthermore, an increase in B lowers $\widehat{T}$ and thus substantially affects the odds of these workers being employed.

Interestingly, because the value of $\widehat{T}$ is decreasing in B, welfare may rise if unemployment insurance is introduced. This follows from the fact that the market solution implies an inefficiently high level of $\widehat{T}$. Thus our model can be used to rationalize unemployment insurance, although in general the social optimum cannot be reached with unemployment insurance.

5.2 Retirement Age

Now we discuss an increase in retirement age T. If T is increased by an amount ΔT, $\widehat{T}$ will increase by the same amount. Following Proposition 4 the expression $\frac{1}{u(\widehat{T})} \int_0^{\widehat{T}} u(\tau) e^{-(b+r+a\gamma)}$ becomes larger, and thus the equilibrium values for a and θ are also higher. As a consequence, the employment rate $1/T \int_0^T \mu_E(\tau)$ is also higher.

It is interesting to note that a reduction in the period needed for schooling or education would have an identical effect if there was no effect on the productivity of workers.

5.3 Hiring Subsidy

A hiring subsidy is usually modelled as a decrease in hiring costs k (cf. Mortensen and Pissarides (1999a)). Such a decrease in k effects an increase in $\widehat{T}$. Thus workers stop searching at a higher age. Following Proposition 4, the expected value of a new match unequivocally increases. Thus labor-market tightness θ and the rate a, which determines how quickly workers find jobs, will decrease. Again it is interesting that the impact of a hiring subsidy on a and on employment is stronger once the fact is taken into account that workers do not live forever. For reasons analogous to those laid out in subsection 5.1, a hiring subsidy has the strongest effect for very young workers ($\tau = 1/a$) and for old workers ($\tau = \widehat{T}$).

5.4 Income Tax

Let us now discuss the impact of income tax, i.e., a situation where the state collects a fraction $\mathcal{T}$ of the wage and a fraction of the firms' profits, which are given by output minus wages, recruiting costs c, and training costs k. Thus we have to replace $c \rightarrow (1-\mathcal{T})c$, $k \rightarrow (1-\mathcal{T})k$, $A \rightarrow (1-\mathcal{T})A$ in equations (26) and (27). It is easy to see that this works like an increase in B, which we have already discussed.[15] An increase in income tax therefore lowers $\widehat{T}$, the rate a, and overall employment.

6 Example

In this section we discuss an example illustrating our findings. The parameters for our example are given in Table 1. Note that these parameters imply that an existing match is severed by an exogenous shock after $1/0.1 = 10$ years on average. The value of T implies that workers can work for a maximum of 45 years.

unemployment benefits B	5.00
productivity level A	10.00
discount rate r	0.04
workers' bargaining power γ	0.50
breakup rate for matches b	0.10
retirement age T	45.00
costs of maintaining a vacancy c	1.00
costs of starting a match k	10.00
matching function $m(u,v)$	$\sqrt{u}\sqrt{v}$

Table 1: Values for the parameters

In Table 2 we compare the results of our model with a model of workers with finite lives. Our earlier claim that a is always a bit lower once one takes the fact into account

[15] We have assumed that B cannot be taxed by the government.

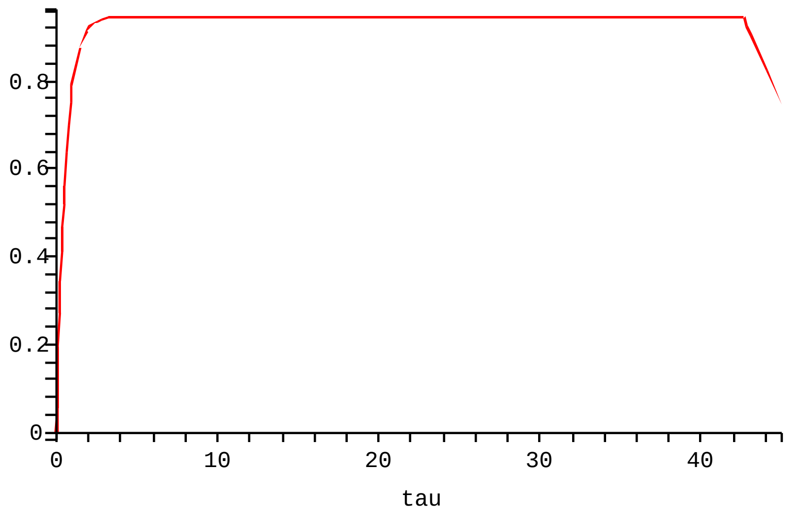

Figure 2: Employment as a function of τ

that workers live for a finite period of time can now be verified. Additionally, the fraction of employed workers is also lower compared to the more standard case. This can be partially ascribed to the fact that all workers out of a job who are older than $\widehat{T} = 42.65$ will not search for a new job.

Our model implies that employment will increase for very young workers and decline again for old workers. Young workers start without a job and, over time, may find a job. Old workers with $\tau > \widehat{T}$ may lose their jobs, when they are separated at rate b, but they will never find a job subsequently. To illustrate this finding, Figure 2 shows the fraction of employed workers as a function of worker age. Interestingly, our model implies a similar pattern to the one found in Figure 1 for the United States and the group of EU-15 states.

	finite lives	infinite lives
a	1.74	1.76
$\widehat{T}$	42.65	-
employed	92.90%	94.63%
unemployed, searching	6.28%	5.37%
unemployed, not searching	0.82%	-

Table 2: Results

In Table 3 we examine the quantitative impact of a 10% increase in unemployment benefits. We see that the increase in unemployment insurance induces workers to stop searching for a job at a lower age, i.e. $\widehat{T}$ decreases. It is apparent that the

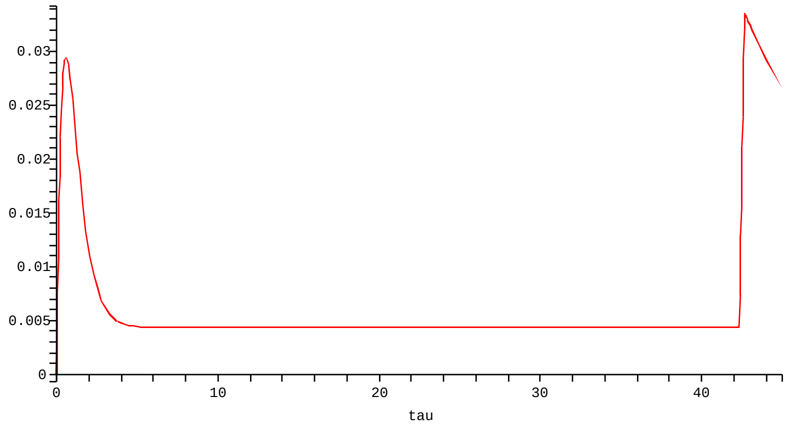

Figure 3: The change in employment as a result of a 10% increase in unemployment benefits.

impact on a and on employment is substantially larger for workers with finite lives than for workers living indefinitely. Furthermore, the increase in unemployment insurance affects different age groups differently. This is displayed in Figure 3, which confirms the statement made in Proposition 5 that the effect of an increase in B is strongest for very young and very old workers. This is similar to the pattern found in Figure 1, where in the United States and the EU-15 countries the percentage of employed workers differs only for very young and very old workers.

relative change in	finite lives	infinite lives
a	-7.81%	-7.73%
$\widehat{T}$	-0.74%	-
# employed	-0.71%	-0.44%
# unemployed, searching	7.24%	7.89%
# unemployed, not searching	25.33%	-
# jobless, overall	9.32%	7.89%

Table 3: Impact of a 10% increase in unemployment benefits.

7 Conclusions

Our model of workers with finite lives has provided us with several insights. We have shown that old workers generally search for longer than the social optimum, because they do not consider the negative externality that an increased average age for newly hired workers has on firms. In this respect, one might wonder why firms do not screen workers according to age. First, discrimination due to age, i.e. offering vacancies to workers only up to a certain age, may be illegal. Second, any match with a worker younger than $\widehat{T}$ creates positive future profits for a firm, although the value of the match may not exceed the sunk costs which stem from posting the vacancy. Thus refraining from hiring comparably old workers is not time-consistent.

Note that our findings about efficiency depend on the fact that the unemployment benefits B represent the output of home production or the disutility of work. However, if these benefits were transfers from the state, it would be possible for workers to quit searching for a job at an earlier state than the social optimum.

We have also shown that the steady state is not necessarily unique even for a model which would have a unique solution if workers lived indefinitely. Multiple equilibria may occur particularly if the bargaining power of workers is low. Low bargaining power on the workers' part implies that for firms the value of a match may be increasing in a and thus θ, as a higher value of a implies a lower average age for newly hired workers. Because the average costs accruing before a vacancy is filled will also increase in θ, multiple equilibria may occur.

Our model has another implication: Wages are approximately constant for young workers. For old workers who are close to retirement, they are lower than for young workers of the same productivity level, since matches become less valuable as workers approach retirement. The wage behavior in our model hinges on our assumption that wages are determined by Nash bargaining and are constantly adjusted. If wages were agreed upon when the match is formed and then remained fixed, they would decrease with the age of the worker when she was hired. Thus we would obtain a similar age pattern for wages as on average older workers would earn less than younger workers of the same productivity. If we introduced an efficiency wage mechanism into our model, then wages would be independent of age (unless the costs of effort were age-dependent). We demonstrate this claim in the Appendix. It is also possible that implicit contracts, an increased bargaining power on the part of those workers who have worked for a long time, would ensure that wages rise when workers get older. Thus the analysis of wages as a function of age might in principle allow for discrimination between different wage-setting mechanisms.

It is crucial to note that we identify an effect of age on wages that results only from the fact that workers will retire at some age. We abstract completely, e.g., from the experience workers gain over time. In reality, the productivity of young workers, in particular, will increase as they get older, which will result in wage increases. While in our model wages decrease with age for older workers of a specified productivity level, the average wage may well increase with age if productivity increases with tenure. A more detailed analysis shows that while the wage decreases over time for a given productivity level, the average wage may nevertheless increase with worker age.[16]

Our model implies that the employment rate as a function of worker age is increasing for young workers and decreasing for old workers. This is a realistic feature, which can be found, e.g. in the United States and EU-15 (compare Figure 1). Interestingly, the differences in employment rates between the United States and EU-15 found in the data are substantial for very young and very old workers, whereas the employment rates are similar for workers of intermediate age. This is consistent with our analysis, as our model predicts that the bulk of employment losses created by generous unemployment benefits or comparably high taxes affects very young and very old workers.

We have also shown that the impact of policy measures on the rate at which unemployed workers find jobs and on employment is generally stronger when one recognizes the fact that workers can only work up to certain age. The change in employment as a consequence of policy measures is usually much stronger for workers close to retirement or for workers who are very young. Hence, when estimating the effects of policy by calibrated models it might be worthwhile to take account of the fact that workers can only work up to a certain age.

[16] A more detailed analysis of this case is available on request.

A Derivation of $\mathbf{w}^-(\tau)$

Here we derive the expression for $w^-(\tau)$ given in the main text. Equation (6) can be transformed into

$$w^-(\tau) = A - (b+r)V_F^-(\tau) + \dot{V}_F^-(\tau). \tag{35}$$

If we apply $V_F^-(\tau) = (1-\gamma)X^-(\tau)$, this can be rewritten as

$$w^-(\tau) = A + (1-\gamma)\left[-(b+r)X^-(\tau) + \dot{X}^-(\tau)\right]. \tag{36}$$

Equation (20) can be rearranged to

$$-(b+r)X^-(\tau) + \dot{X}^-(\tau) = -\Delta A - a\gamma k + a\gamma X^-(\tau). \tag{37}$$

By combining (36) and (37) we obtain

$$w^-(\tau) = A + (1-\gamma)\left(-\Delta A - a\gamma k + a\gamma X^-(\tau)\right). \tag{38}$$

By inserting the solution for $X^-(\tau)$, which is given by Equation (21), we obtain

$$w^-(\tau) = A - (1-\gamma)\left((b+r)\frac{\Delta A + a\gamma k}{b+r+a\gamma} + a\gamma\frac{\Delta A - (b+r)k}{b+r+a\gamma}e^{-(b+r+a\gamma)(\widehat{T}-\tau)}\right). \tag{39}$$

□

B Proof of Proposition 1

It suffices to show that Equation (26) must hold. Note that (22) and (25) can be combined to

$$\frac{c}{\alpha} = (1-\gamma)\cdot\frac{\Delta A - (b+r)k}{b+r+a\gamma}\cdot\left(1 - \frac{1}{u}\int_0^{\widehat{T}} \mu_U^-(\tau)e^{-(b+r+a\gamma)(\widehat{T}-\tau)}\,d\tau\right). \tag{40}$$

Partial integration yields

$$\int_0^{\widehat{T}} \mu_U^-(\tau)e^{-(b+r+a\gamma)(\widehat{T}-\tau)}\,d\tau = u(\widehat{T}) - (b+r+a\gamma)\int_0^{\widehat{T}} u(\tau)e^{-(b+r+a\gamma)(\widehat{T}-\tau)}\,d\tau. \tag{41}$$

Thus we obtain

$$\frac{c}{\alpha} = (1-\gamma)\cdot(\Delta A - (b+r)k)\int_0^{\widehat{T}} \frac{u(\tau)}{u(\widehat{T})}e^{-(b+r+a\gamma)(\widehat{T}-\tau)}\,d\tau. \tag{42}$$

□

C Proof of Proposition 2

We show that Equation (26) always has a solution for θ. Note that for $\theta \to \infty$, $\alpha(\theta) = 0$. This implies that the left-hand side of the equation goes to infinity. Let us now consider the right-hand side. If $\theta \to \infty$, then $a(\theta) = \infty$. As a consequence, the value of a new match converges to zero ($X_L^-(\tau) - k = 0$). Hence the right-hand side of (26) goes to zero.

Now consider $\theta \to 0$. This implies that the left-hand side of the equation converges to zero because $\alpha(\theta) \to \infty$. For $\theta \to 0$, we obtain $a(\theta) \to 0$. Consequently, the value of a match $X_L^-(\tau) - k$ converges to a finite number for each τ. This implies that the right-hand side of (26) converges to a finite number. Hence, by the mean-value theorem, we have established that a solution to Equation (26) exists.

$\square$

D Derivation of the Efficient Solution

In the following we examine the efficient solution a social planner would choose who tries to maximize the discounted sum of output. This will enable us to examine whether the solution derived in Section 3 is efficient. We will start by considering discrete time, then, taking the limit $dt \to 0$ to proceed to continuous time.

Let $\overrightarrow{\mu}_{E,t}$ denote the vector describing the fraction of workers of a certain age who are employed at time t, i.e. the first component of $\overrightarrow{\mu}_{E,t}$ denotes the fraction of workers at age 0 at time t. This component will be labelled $\overrightarrow{\mu}_{E,t,0}$. According to our assumption that all workers are unemployed when entering the labor market, this component always amounts to zero. The last component of $\overrightarrow{\mu}_{E,t}$, which is denoted by $\overrightarrow{\mu}_{E,t,T}$, corresponds to the respective fraction for workers of age T.

We introduce the following definition for $x \in \{E, U, 0\}$:

$$|\overrightarrow{\mu}_{x,t}| := \frac{1}{|\mathbf{I}|} \sum_{\tau \in \mathbf{I}} \mu_{x,t,\tau}, \tag{43}$$

where $\mathbf{I} = \{0, dt, 2dt, ..., T-dt, T\}$ and $|\mathbf{I}|$ is the number of elements in $\mathbf{I}$.[17] For example,

[17] We implicitly assume that there exists a number n such that $n\,dt = T$.

$|\overrightarrow{\mu}_{E,t}|$ represents the overall number of employed workers. We also introduce

$$|\overrightarrow{\mu}_{x,t}|^- := \frac{1}{|\mathbf{I}|}\sum_{\tau\in\mathbf{I}^-}\mu_{x,t,\tau}, \tag{44}$$

where $\mathbf{I}^- = \{\tau \in I : \tau \leq \widehat{T}\}$. Finally we define $\overrightarrow{1}$ as the $|\mathbf{I}|$-dimensional vector which has only entries of 1. For example $|\overrightarrow{1} - \overrightarrow{\mu}_{E,t}|^-$ equals u and $|\overrightarrow{1} - \overrightarrow{\mu}_{E,t}|^-\theta$ equals v.[18]

Now we can write the Bellman equation for the social planner as

$$\begin{aligned} &Y(\overrightarrow{\mu}_{E,t}) \\ &= \max_{\theta,\widehat{T}} \frac{1}{1+r\,dt}\Big[\Delta A|\overrightarrow{\mu}_{E,t}|\,dt + B\,dt \\ &\qquad - \left|\overrightarrow{1} - \overrightarrow{\mu}_{E,t}\right|^- \theta c\,dt - a\left|\overrightarrow{1} - \overrightarrow{\mu}_{E,t}\right|^- k\,dt + Y(\overrightarrow{\mu}_{E,t+dt})\Big]. \end{aligned} \tag{45}$$

Note that Y depends on the whole age distribution of employed workers. The overall output per period is given by the output produced by employed workers $A\,dt\,|\overrightarrow{\mu}_{E,t}|$ and the benefits created by workers not employed $|1-\overrightarrow{\mu}_{E,t}|B\,dt$. Together these terms correspond to $\Delta A|\overrightarrow{\mu}_{E,t}|\,dt + B\,dt$. We also have to take into account the costs for vacancies $cv = \left|\overrightarrow{1} - \overrightarrow{\mu}_{E,t}\right|^- \theta c$ and the training costs $a\left|\overrightarrow{1} - \overrightarrow{\mu}_{E,t}\right|^- k\,dt$. The social planner has two decision variables, namely θ and $\widehat{T}$. He also has to take into account the following restriction for $\tau < \widehat{T}$.

$$\mu^-_{E,t+dt,\tau} = \mu^-_{E,t,\tau-dt} - b\,dt\,\mu^-_{E,t,\tau-dt} + a\,dt(1-\mu^-_{E,t,\tau-dt}).$$

This condition is easy to interpret. $\mu^-_{E,t+dt,\tau}$ is given by the fraction of workers employed in the previous period t who were a bit younger $(\mu^-_{E,t,\tau-dt})$. However, some of these matches were separated $(-b\,dt\,\mu^-_{E,t,\tau-dt})$, and some formerly unemployed workers have found work $(a\,dt(1-\mu^-_{E,t,\tau-dt}))$.

For $\tau > \widehat{T}$, the social planner has to take account of the fact that unemployed workers do not search for a job. Thus the fraction of employed workers is a decreasing function of worker age.

$$\mu^+_{E,t+dt,\tau} = \mu^+_{E,t+dt,\tau-dt} - b\,dt\,\mu^+_{E,t+dt,\tau-dt}.$$

By the contraction mapping theorem the solution to this optimization problem is unique.[19] We surmise that this unique solution Y is a linear function of the vector $\overrightarrow{\mu}_{E,t}$. Thus we can write Y as

$$Y(\overrightarrow{\mu}_{E,t}) = \kappa + \overrightarrow{x}\,\overrightarrow{\mu}_{E,t}. \tag{46}$$

[18]Recall that θ (labor-market tightness) was defined to be $\theta = \frac{v}{u}$.
[19]Compare, e.g., Mortensen and Pissarides (1999a).

Note that the vector $\overrightarrow{x}$ describes the social value of matches. The single components of $\overrightarrow{x}$ contain the social value of matches with workers of a certain age.[20]

If we apply the envelope theorem, we obtain for $\tau < \widehat{T}$:

$$x_\tau^- = \frac{1}{1 + r\,dt}\left[\Delta A\,dt + c\theta\,dt + ak\,dt + x_{\tau+dt}^-(1 - b\,dt - a\,dt)\right].$$

This is equivalent to the following "asset-price" equation:

$$rx_\tau^- = \Delta A + c\theta + ak - (a+b)x_{\tau+dt}^- + \frac{x_{\tau+dt}^- - x_\tau^-}{dt}. \tag{47}$$

If we take the limit $dt \to 0$, this difference equation becomes a differential equation:

$$(a + b + r)x^-(\tau) = \Delta A + c\theta + ak + \dot{x}^-(\tau). \tag{48}$$

For $\tau > \widehat{T}$, the envelope theorem yields the difference equation

$$rx_\tau^+ = \Delta A - bx_{\tau+dt}^+ + \frac{x_{\tau+dt}^+ - x_\tau^+}{dt},$$

which implies

$$(b + r)x^+(\tau) = \Delta A + \dot{x}^+(\tau).$$

For $d\tau \to 0$. It can readily be verified that for $\tau > \widehat{T}$ the solution is given by

$$x^+(\tau) = \frac{\Delta A}{b + r}\left(1 - e^{-(b+r)(T-\tau)}\right). \tag{49}$$

We can now use (48) to derive

$$x^-(\tau) = \int_\tau^{\widehat{T}} (\Delta A + c\theta + ak)\, e^{-(a+b+r)(\widehat{T}-\tau')}d\tau' + e^{-(a+b+r)(\widehat{T}-\tau)}x^+(\widehat{T}). \tag{50}$$

Optimization with respect to θ yields the first-order condition

$$\left|\overrightarrow{1} - \overrightarrow{\mu}_{E,t}\right|^- c\,dt + \left|\overrightarrow{1} - \overrightarrow{\mu}_{E,t}\right|^- a'(\theta)k = a'(\theta)\sum_{\tau \in \mathbf{I}^-} x_\tau \left(1 - \mu_{E,t+dt,\tau-dt}^-\right).$$

If we take the limit $dt \to 0$ and thus switch to continuous time, we obtain

$$c = a'(\theta)\left(\frac{1}{u}\int_0^{\widehat{T}} \left(x^-(\tau) - k\right)\mu_{U,t}(\tau)d\tau\right). \tag{51}$$

[20]We apply the boundary condition $x_T^+ = 0$. This condition is arbitrary and only affects the value of the constant κ. We apply this condition to make the analysis similar to the one in Section 3.

This condition states that the socially optimal choice of θ must balance the cost of an additional vacancy (on the left-hand side of the equation) and the benefits of an increased flow of new matches (on the right-hand side).

Now we compute the difference between the maximand of (45) for $\widehat{T}$ and for $\widehat{T} - dt$. Thus the optimization with respect to $\widehat{T}$ yields the second first-order condition

$$-(1 - \mu_{E,t}(\widehat{T}))(\theta c\, dt + ak\, dt) + (1 - \mu_{E,t}(\widehat{T}))a\, dt\, x^{+}_{\widehat{T}+dt} = 0,$$

which is equivalent to

$$a\left(x^{+}(\widehat{T}) - k\right) = \theta c. \tag{52}$$

Note that an increase in $\widehat{T}$ involves costs per the unemployed worker affected. These costs are given by θc because additional vacancies must be created to keep θ constant. On the other hand, an increase in $\widehat{T}$ may create potential benefits $x(\widehat{T}) - k$ because unemployed workers may find a job at rate a.

Thus the optimal value of $\widehat{T}$ is then given by

$$\widehat{T}_{Opt} = \frac{1}{b+r}\ln\left(1 - \frac{(b+r)(k+\theta c/a)}{\Delta A}\right) + T. \tag{53}$$

Using Equation (52), we can now rearrange (50) to

$$x^{-}(\tau) - k = \frac{\Delta A + c\theta - (b+r)k}{a+b+r} - \frac{\Delta A - (b+r)(c\theta/a + k)}{a+b+r}e^{-(a+b+r)(\widehat{T}-\tau)}. \tag{54}$$

Equations (51), (53) and (54) characterize the social optimum.

□

E Proof of Proposition 5

Note that a marginal increase in unemployment insurance B only has an impact on $\mu^{-}_{E}(\tau)$ through its impact on a. It is straightforward to show that the cross derivative of $\mu^{-}_{E}(\tau)$ with respect to a and to τ is given by

$$\frac{d^2\,\mu^{-}_{E}(\tau)}{d\tau\, da} = (1 - a\tau)e^{-(a+b)\tau}, \tag{55}$$

which is equal to zero for $\tau = 1/a$. Because $\frac{d^3\,\mu^{-}_{E}(\tau)}{d^2\tau\, da} > 0$, we can conclude that $\frac{d\mu^{-}_{E}(\tau)}{da}$ has a local minimum at $\tau = 1/a$. Hence a marginal increase in B effects a marginal decrease in a, which has its strongest impact on the unemployment rate for workers of age τ.

The second part follows from the fact that $\frac{\mu_E^+(\tau)}{da}$ is highest for $\tau = \widehat{T}$.

□

F Proof of Proposition 4

Now we show that $\frac{1}{u(\widehat{T})}\int_0^{\widehat{T}} u(\tau)e^{-\kappa(\widehat{T}-\tau)}\,d\tau$ is increasing in $\widehat{T}$ where we have introduced $\kappa := \gamma a + b + r$. The derivative is given by

$$\begin{aligned} &\frac{d}{d\widehat{T}}\frac{1}{u(\widehat{T})}\int_0^{\widehat{T}} u(\tau)e^{-\kappa(\widehat{T}-\tau)}\,d\tau \\ = \;& \int_0^{\widehat{T}} u(\tau)e^{-\kappa(\widehat{T}-\tau)}\,d\tau\left(-\frac{u'(\widehat{T})}{(u(\widehat{T}))^2}\right) \\ &+\frac{1}{u(\widehat{T})}\left[u(\widehat{T}) - \kappa\int_0^{\widehat{T}} u(\tau)e^{-\kappa(\widehat{T}-\tau)}\,d\tau\right]. \end{aligned}$$

This expression is positive if

$$(u(\widehat{T}))^2 - \int_0^{\widehat{T}} u(\tau)e^{-\kappa(\widehat{T}-\tau)}\,d\tau\left(\kappa u(\widehat{T}) + u'(\widehat{T})\right) > 0. \tag{56}$$

It is very tedious but straightforward to show that this is equivalent to

$$\begin{aligned} \frac{1}{\kappa^2}\Bigg\{ & b^2(1 - e^{-\kappa T} - \kappa T e^{-\kappa T}) + bae^{-(a+b)T}(1 - e^{-\kappa T}) \\ & +\kappa a e^{-(a+b+\kappa)T}\left[1 + (1+bT)\frac{(a+b)e^{\kappa T} - \kappa e^{(a+b)T}}{\kappa - a - b}\right]\Bigg\} > 0. \end{aligned} \tag{57}$$

This inequality holds because the following three inequalities hold:

1. $(1 - e^{-\kappa T} - \kappa T e^{-\kappa T}) > 0$: This follows from $x > \ln(1+x)\ \forall x > 0$.
2. $bae^{-(a+b)T}(1 - e^{-\kappa T}) > 0$: This is a consequence of the fact that $\kappa > 0$.
3. $1 + (1+bT)\frac{(a+b)e^{\kappa T} - \kappa e^{(a+b)T}}{\kappa - a - b} > 0$: This follows from two observations. First the expression is zero for $T = 0$. Second it is monotonously increasing in T because $\frac{(a+b)e^{\kappa T} - \kappa e^{(a+b)T}}{\kappa - a - b}$ is increasing in T both for $\kappa > a + b$ and for $\kappa < a + b$.

□

G Efficiency Wages as a Function of Age

Now we demonstrate in a model based on Shapiro and Stiglitz (1984) wages do not depend on age as long as the disutility of effort is age-independent. Assume that there is only one productivity level A, which corresponds to the case $p \to \infty$. A worker can either exert effort at a cost $\overline{e}$. Then the worker will produce A. She can also shirk. Then no output is produced, and the worker does not incur the costs $\overline{e}$. Each shirker is detected at a rate s and fired. In equilibrium the firm chooses the wage so as to make the worker indifferent between shirking and exerting effort. The following equations must hold:

$$
\begin{aligned}
rV_S(\tau) &= w(\tau) + (s+b)\left(V_U(\tau) - V_S(\tau)\right) + \dot{V}_S(\tau) && (58)\\
rV_E(\tau) &= w(\tau) - \overline{e} + b\left(V_U(\tau) - V_E(\tau)\right) + \dot{V}_E(\tau) && (59)\\
rV_U(\tau) &= B + a\left(\max\{V_E(\tau), V_S(\tau)\} - V_U(\tau)\right) + \dot{V}_U(\tau), && (60)
\end{aligned}
$$

where the subscript S denotes shirking and the other variables have the same meaning as in the main text.

By using $V_E(\tau) = V_S(\tau)$ these equations can be written as

$$
\begin{aligned}
rV_E(\tau) &= w(\tau) + (s+b)\left(V_U(\tau) - V_E(\tau)\right) + \dot{V}_E(\tau) && (61)\\
rV_E(\tau) &= w(\tau) - \overline{e} + b\left(V_U(\tau) - V_E(\tau)\right) + \dot{V}_E(\tau) && (62)\\
rV_U(\tau) &= B + a\left(V_E(\tau) - V_U(\tau)\right) + \dot{V}_U(\tau). && (63)
\end{aligned}
$$

If we subtract the second equation from the first we obtain

$$\overline{e} = s\left(V_E(\tau) - V_U(\tau)\right). \qquad (64)$$

This equation states that the benefits of shirking, which stem from avoiding the costs $\overline{e}$, must equal the costs of shirking, which result from an increased probability of being fired.

Subtracting the third from the second equation yields

$$r\left(V_E(\tau) - V_U(\tau)\right) = w(\tau) - \overline{e} - B - (a+b)\left(V_U(\tau) - V_E(\tau)\right) + \dot{V}_E(\tau) - \dot{V}_U(\tau). \qquad (65)$$

If we insert (64) into (65), we obtain after some rearrangements

$$w(\tau) = \overline{e} + \frac{r+a+b}{s}\overline{e}. \qquad (66)$$

This is the standard equation we would also obtain for workers with infinite lives.[21] In particular, it is important to note that wages do not depend on time. If the costs of effort $\overline{e}$ increased over time, then we would obtain wages that increase with age. Thus efficiency wages imply a completely different pattern for wages as a function of age than for Nash-bargained wages.

$\square$

[21]Compare e.g. Romer (2005).

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