Modelling Alpha-Opportunities Within the CAPM*

Anke Gerber [a] Thorsten Hens [a,b]

July 2006

Abstract

We consider a simple CAPM with heterogenous expectations on assets' mean returns and homogenous expectations on the covariance of returns. In this model alpha-opportunities naturally arise in a financial market equilibrium. We show that that the hunt for alpha-opportunities is a zero-sum game and that alpha-opportunities erode with the assets under management. Moreover, it is shown that a positive alpha is not necessarily a good criterion for the choice between active and passive investment. Finally, we argue that the standard CAPM with homogenous beliefs can be seen as the long run outcome of our model when investors' expectations are endogenous.

Keywords: CAPM, heterogenous beliefs, active and passive investment.
JEL-Classification: G11, G12, G14.

---

*Financial support by the University Research Priority Programme "Finance and Financial Markets" of the University of Zurich and the national center of competence in research "Financial Valuation and Risk Management" is gratefully acknowledged. The national centers in research are managed by the Swiss National Science Foundation on behalf of the federal authorities.

[a]Swiss Banking Institute, University of Zurich, Plattenstrasse 32, CH-8032 Zurich, Switzerland. Email: agerber@isb.unizh.ch, thens@isb.unizh.ch.

[b]Norwegian School of Economics and Business Administration, Helleveien 30, N-5045, Bergen, Norway.

1 Introduction

The Capital Asset Pricing Model, CAPM, is a rich source of intuition and also the basis for many practical financial decisions. The asset pricing implication of the CAPM is the security market line, SML, according to which the excess return of any asset over the risk free rate is proportional to the excess return of the market portfolio over the risk free rate. The proportionality factor is the *beta*, i.e. the covariance of the asset's return to the return of the market portfolio divided by the variance of the market portfolio. The beta is the only risk factor that is rewarded according the CAPM. Hence an investor requiring a high expected return will have to accept a high beta. Some investors however want to achieve more. They claim to be able to achieve positive deviations of expected returns over those given by the SML. Those deviations of returns are referred to as Jensen's alpha or short as the "*alpha*." Indeed the alpha is nowadays a common term in the finance jargon. Hedge Funds for example consider themselves to be alpha generating strategies; many of them use the term "alpha" in their marketing brochures and some of them even as part of their name.

While many opinion leaders in the world of finance claim that the existence of alpha contradicts the validity of the CAPM, we argue in this paper that a simple extension of the CAPM towards heterogenous beliefs is already able to explain the alpha in a financial market equilibrium. The extension we use goes back to one of the CAPMs with heterogenous beliefs suggested by Lintner (1969). Because the most general CAPM with heterogenous beliefs becomes intractable we will use one of the Lintner-CAPMs in which investors have heterogenous beliefs over expected returns while they agree on the covariances of returns. The assumption of homogeneous covariance expectations is frequently used in the literature, but, of course, is a severe restriction to investors' beliefs.[1] However, since the main purpose of the CAPM is to organize ideas, this assumption seems justified because it keeps the CAPM tractable. Moreover, many practitioners do portfolio allocations using historic covariances while adjusting historic means

[1]The famous model of Brock and Hommes (1998), for example, is based on mean-variance optimizing agents that have heterogenous beliefs on expected returns but agree on covariances. This model does, however, only have one risky asset.

to get reasonable expected returns. Finally, in the CAPM means are first order effects while covariances capture second order effect. Hence mistakes in means hurt the investor more than equally sized mistakes in covariances (cf. Chopra and Ziemba (1993)).

Our first result derives the security market line of the CAPM as an aggregation result without using the unrealistic two-fund-separation property. The security market line turns out to hold with respect to *average* expectations, where the weight of each investor is proportional to the investor's wealth divided by his risk aversion. Hence the more wealthy and the less risk averse an investor the more do his expectations determine the average expectations. In particular the often observed feature of underdiversification (see, for example, Odean (1999), Goetzmann and Kumar (2005) and Polkovnichenko (2005)) can well be compatible with optimal portfolio choice. If an investor has superior information then underdiversification can even be necessary to outperform the market! In our model alpha-opportunities can be explained as a feature of financial market equilibria. The further the average expected returns deviate from the true returns the higher the alpha-opportunities. Moreover we can show that alpha-opportunities erode with the assets under management, which is a feature that has been observed for many active portfolio managers, as for example for hedge funds (cf. Getmansky (2004) and Agarwal, Daniel, and Naik (2005)). In our model this important feature has a very simple explanation. The more wealth a strategy acquires the more it resembles the market portfolio, which, by definition, has an alpha of zero. Note that our model gives an equilibrium explanation of this feature that does not need to refer to any ad-hoc ideas of a production function for alpha opportunities (cf. Berk and Green (2004)). Moreover, in our model the hunt for alpha-opportunities is a zero-sum game. If some investor generates a positive alpha there must be some other investor earning a negative alpha. Hence the ease to generate alpha opportunities depends on the sophistication of the other investors in the market. This feature may explain why hedge funds could generate very high returns during the stock market bubble of the turn of the millennium in which many unsophisticated investors took active bets. After the bubble burst, many unsophisticated investors left the market due to frustration and hedge fund returns decreased.

We extend our model by endogenizing agents' information by allowing them to be either passive, in which case they invest according to the average expectation embodied in the market returns, or to be active, in which case they can acquire superior information at some cost. In our model we show that the decision of being active or passive depends on the efficiency of the market, the quality of the investor's belief, his degree of risk aversion and of course the costs for being active. An investor is more inclined to be active the less efficient the market is, the better his information and the less risk averse he is. By contrast, it can be shown that expecting a positive alpha is not necessarily a good criterion for becoming active. We give simple examples pointing out that expecting a positive alpha from the active strategy is neither a necessary nor a sufficient condition for becoming active. In our model, delegating active investment to portfolio managers only makes sense if the performance fee increases with the skill of the portfolio manager and is bounded above by some function of the degree of inefficiency of the market. Our model provides new measures for both of these components. Finally, in our model it turns out that a market in which some investors acquire information to be active while the others get the average information for free from market prices cannot be a stable outcome. Moreover, all investors being passive may also not be an outcome that is stable with respect to information acquisition if the average expectation is far from the true returns. This result resembles the well know result of Grossman and Stiglitz (1980) on the impossibility of informational efficient markets. Accounting for this stability requirement, the standard CAPM with homogenous and correct beliefs can be seen as the long run outcome of our model. Hence we can argue that alpha opportunities can arise in a financial market equilibrium as a reaction to a non-stationarity like an exogenous shock (invention of the railway, the mass production or the internet) but under sufficiently stationary exogenous conditions alpha-opportunities will vanish.

Our results give a common framework for many phenomena that have been discussed in the literature. Besides being able to address alpha-opportunities in a simple equilibrium framework, we can explain underdiversification, the erosion of alpha-opportunities as assets under management increase, and the structure of performance fees for active management. Moreover, our simple model gives a foundation of more applied research on active management like the one of Grinold and Kahn (2000) and Black and Litterman (cf. Litterman (2003)). Our model provides a common ground for these two approaches whose methodologies seem to be in contradiction. While Grinold and Kahn (2000) argue for active portfolio management based on the mean-variance framework of Markowitz, Black and Litterman argue for active portfolio management based on the security market line. Black and Litterman assume that the security market line is a "center of gravity" towards which the financial markets tend over time. Hence an active Black-Litterman investor goes short in those assets that have realized a positive alpha because he infers from this that in the next period the return will most likely be decreasing. Our model gives support to this view since taking account for the optimal information acquisition in the long run all alphas will erode. Our approach can also accommodate active portfolio management in the sense of Grinold and Kahn. As we show below, optimal mean-variance portfolios must lie on a security line which is the security market line in which market expectations have been replaced by individual expectations. The security market line is then obtained by the aggregation of these individual security lines. An active mean-variance investor à la Grinold and Kahn "sees" alpha opportunities because he holds a belief of expected returns that deviates from the average belief of the investors expressed in the security market line.

Of course we do not claim that our simple model can explain all features of active management. In particular some features related to hedge funds, as for example higher order returns, lead out of the mean-variance framework. However, since a simple CAPM with heterogenous beliefs carries us quite far in the understanding of many important features of active management this framework can give a first intuition for what active management is about.

The rest of the paper is organized as follows. The next section gives a formal description of the CAPM with heterogenous beliefs and it derives the aggregation result of the SML. Then we allow the investors to choose whether they do active or passive portfolio management. In that section we derive the main criterion for active portfolio management based on the measures of market efficiency and the skill of the active managers. Furthermore we show which structure fees for active management should have. Thereafter we consider the alpha. We show the zero-sum property of alpha opportunities and that they erode with increasing asset under management. Moreover, we question whether expecting a positive alpha is an appropriate criterion for becoming active.

2 A CAPM with Heterogenous Beliefs

We consider a two periods financial market model with dates $t = 0, 1$, and $k = 0, 1, \ldots, K$ assets with payoffs $A^k \in \mathbb{R}$ in $t = 1$.[2] Asset $k = 0$ is riskless and its return is denoted by R_f. Assets $k = 1, \ldots, K$, are risky, i.e. A^k is a random variable with $\sigma^2(A^k) \neq 0$. We assume that $\sigma^2\left(\sum_{k=1}^{K} \theta_k A^k\right) \neq 0$ for all $0 \neq \theta \in \mathbb{R}^K$, i.e. there are no redundant assets. Moreover, we assume that A^k has a finite mean $\mathbb{E}(A^k)$ for all $k = 1, \ldots, K$. Let $S = \left(\mathrm{COV}(A^k, A^l)\right)_{k,l=1,\ldots,K}$ be the covariance matrix of asset payoffs, where $\mathrm{COV}(A^k, A^l)$ is finite for all $k, l = 1, \ldots, K$. Since there are no redundant assets it follows that S is positive definite.

The price of asset $k, k = 1, \ldots, K$, in $t = 0$ is denoted by q^k. Assume for the moment that $q^k \neq 0$ for $k = 1, \ldots, K$.[3] Then let $R^k = A^k/q^k$ denote the return of asset k. By $\hat{\mu}_k = \mathbb{E}(A^k)/q^k$ we denote the expected return of asset $k, k = 1, \ldots, K$, and by $\mathrm{COV} = \left(\mathrm{COV}(R^k, R^l)\right)_{k,l=1,\ldots,K}$ we denote the covariance matrix of asset returns.[4] Let $\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_K)$. Observe that $\mathrm{COV} = \Delta S \Delta$, where Δ is the diagonal matrix with $\Delta_{k,k} = 1/q^k$ for all $k = 1, \ldots, K$. Without loss of generality we normalize the supply of all risky assets to 1.

There are I investors. Investor i has initial wealth $w_0^i > 0$ and mean-variance preferences over date 1 returns

$$V^i(\mu, \sigma) = \mu - \frac{\gamma^i}{2}\sigma^2,$$

where $\gamma^i > 0$ measures investor i's risk aversion and μ and σ are the expected return and variance, respectively, of investor i's portfolio. We assume that investors do not know the distribution of asset payoffs but rather hold individual

[2] A^k is the cum-dividend price of asset k in $t = 1$.

[3] Later we derive conditions on the exogenous parameters of our economy which guarantee that in equilibrium $q^k \neq 0$ for all k.

[4] Since investors are assumed to take asset prices q as given, we simplify the notation and do not explicitly write $\hat{\mu}$ and COV as functions of q.

beliefs over expected asset payoffs and the covariance matrix of asset payoffs. More specifically, as it has been justified in the introduction, we assume that investors have heterogenous expectations about the average payoff of the assets, but homogenous and correct expectations about the covariance of payoffs. Let $m_k^i = \mathbb{E}^i(A^k)$ be investor i's expectation about the average payoff of asset k and let $m^i = (m_1^i, \ldots, m_K^i)$. Then $\mu^i = (\mu_1^i, \ldots, \mu_K^i)$ with $\mu_k^i = m_k^i / q^k$ for all k denotes i's expectation about average asset returns and COV is the common covariance matrix of returns.

Investor i solves

$$\max_{\lambda \in \mathbb{R}^K} \quad \lambda^T(\mu^i - R_f e) - \frac{\gamma^i}{2} \lambda^T \mathrm{COV} \lambda, \tag{1}$$

where $e = (1, \ldots, 1) \in \mathbb{R}^K$. The (necessary and sufficient) first order condition for the solution λ^i of (1) is

$$\mathrm{COV}\lambda^i = \frac{\mu^i - R_f e}{\gamma^i}. \tag{2}$$

Given the portfolio of risky assets λ^i investor i invests $\lambda_0^i = 1 - \sum_{k=1}^K \lambda_k^i$ into the riskless asset.

The supply of each asset is normalized to 1 so that q^k denotes the market capitalization of asset k and in equilibrium we have

$$q = \sum_i w_0^i \lambda^i.$$

Hence, from the agent's optimal portfolio choice (2) we obtain that

$$\begin{aligned} q &= \sum_i w_0^i \mathrm{COV}^{-1} \frac{\mu^i - R_f e}{\gamma^i} \\ &= \Delta^{-1} S^{-1} \sum_i \frac{w_0^i}{\gamma^i}(m^i - R_f q). \end{aligned}$$

Solving for equilibrium asset prices q we obtain

$$q = \frac{1}{R_f \sum_i \frac{w_0^i}{\gamma^i}} \left(\sum_i \frac{w_0^i}{\gamma^i} m^i - Se \right), \tag{3}$$

and $q^k \neq 0$ for all k if and only if $\sum_i \frac{w_0^i}{\gamma^i} m_k^i \neq \mathrm{COV}(A^k, Ae)$ for all k. Equation (3) resembles the main result of Lintner (1969) for the version of a CAPM with heterogeneous beliefs but common covariance expectations. However, Lintner (1969) did neither derive the SML nor did he address the alpha.

From now on we assume that in equilibrium $\sum_k \lambda_k^i > 0$, so that $\lambda_0^i < 1$.[5]

The asset pricing implication of the standard CAPM is the security market line, SML, according to which the excess return of any asset over the risk free rate is proportional to the excess return of the market portfolio over the risk free rate. The proportionality factor is the *beta*, i.e. the covariance of the asset's returns to the return of the market portfolio divided by the variance of the market portfolio. In order to derive the security market line for our heterogenous expectations we need to specify how individual expectations are averaged to become the market expectation.

In our model it turns out that the appropriate aggregation rule is to define the average expectation for the return of asset k by $\bar{\mu}_k := \sum_i a^i \mu_k^i$, where

$$a^i = \frac{w_0^i}{\gamma^i} \left(\sum_j \frac{w_0^j}{\gamma^j} \right)^{-1}.$$

Hence, every individual's expectation enters the average expectation proportional to the individual's wealth divided by his risk aversion. Accordingly let $\bar{\mu}^M = \sum_k \lambda_k^M \bar{\mu}_k$ with $\lambda_k^M = q^k / (\sum_l q^l)$ for all k be the average expectation of the market portfolio $R^M = \sum_k \lambda_k^M R^k$. Then we can state the Security Market Line Theorem for average expectations as:

[5]Given (2) and (3) we can derive a condition on the exogenous parameters of our economy that guarantees that in equilibrium $\lambda_0^i < 1$ for all i.

Proposition 2.1 (Security Market Line for Average Expectations) *In equilibrium the risk premium of any asset k is proportional to the risk premium of the market portfolio under average expectations, where the factor of proportionality is given by the covariance of the return of asset k with the market portfolio divided by the variance of the market portfolio:*

$$\bar{\mu}_k - R_f = \underbrace{\frac{\mathrm{COV}(R^k, R^M)}{\sigma^2(R^M)}}_{\beta^{M,k}}(\bar{\mu}^M - R_f), \;\; k = 1, \ldots, K. \tag{4}$$

Proof: In order to derive the security market line for our heterogenous expectations economy we define the share of risky wealth as

$$\bar{\lambda}^i_k := \frac{\lambda^i_k}{1 - \lambda^i_0}$$

for all assets k and all investors i. Using (2) this is equivalent to

$$\mathrm{COV}\bar{\lambda}^i = \frac{\mu^i - R_f e}{\gamma^i(1 - \lambda^i_0)}. \tag{5}$$

Let $w^i_f := (1 - \lambda^i_0)w^i_0$ be the financial wealth investor i invests into risky assets. By our assumption above $w^i_f > 0$ for all i. Let, accordingly,

$$r^i = \frac{w^i_f}{\sum_j w^j_f},$$

be the relative financial wealth invested by i. Then multiplying (5) with r^i and summing over all i gives

$$\mathrm{COV}(R^k, R^M) = \sum_i \frac{r^i}{\gamma^i(1 - \lambda^i_0)}(\mu^i_k - R_f), \;\; k = 1, \ldots, K, \tag{6}$$

where $R^M = \sum_k R^k \lambda^M_k$ and $\lambda^M_k = \sum_i \bar{\lambda}^i_k r^i$ for all k. Hence, R^M is the return of the market portfolio. To see this observe that

$$\lambda^M_k = \sum_i \bar{\lambda}^i_k r^i = \sum_i \frac{\bar{\lambda}^i_k w^i_f}{\sum_j w^j_f} = \frac{q^k}{\sum_l q^l},$$

since in equilibrium $q^k = \sum_i \lambda^i_k w^i_0$. Note that $\sum_l q^l = \sum_l \sum_j \lambda^j_l w^j_0 = \sum_j w^j_0(1 - \lambda^j_0) = \sum_j w^j_f$. Moreover, observe that

$$a^i = \frac{w^i_0}{\gamma^i}\left(\sum_j \frac{w^j_0}{\gamma^j}\right)^{-1} = \frac{r^i}{\gamma^i(1 - \lambda^i_0)}\left(\sum_j \frac{r^j}{\gamma^j(1 - \lambda^j_0)}\right)^{-1}$$

for all i. Hence, (6) is equivalent to

$$\frac{\mathrm{COV}(R^k, R^M)}{\sum_i \frac{r^i}{\gamma^i(1-\lambda_0^i)}} = \bar{\mu}_k - R_f,\ k = 1, \ldots, K. \tag{7}$$

Multiplying with λ_k^M and summing over all k gives

$$\frac{\sigma^2(R^M)}{\sum_i \frac{r^i}{\gamma^i(1-\lambda_0^i)}} = \bar{\mu}^M - R_f, \tag{8}$$

where $\bar{\mu}^M = \sum_k \lambda_k^M \bar{\mu}_k = \sum_i a^i \mu^i(R^M)$, with $\mu^i(R^M) = \sum_k \lambda_k^M \mu_k^i$. Hence, $\bar{\mu}^M$ is the average expectation of the return of the market portfolio. Substituting this into (7) gives the Security Market Line for average expectations. □

Coming back to the individual optimization recall that each investor chooses an investment strategy λ^i that solves (5), i.e.

$$\mathrm{COV}(R^k, \sum_l \bar{\lambda}_l^i R^l) = \frac{\mu_k^i - R_f}{\gamma^i(1-\lambda_0^i)}. \tag{9}$$

Multiplying both sides of (9) with $\bar{\lambda}_k^i$ and summing over all k gives

$$\gamma^i(1-\lambda_0^i)\sigma^2(R^{\bar{\lambda}^i}) = \mu^i(R^{\bar{\lambda}^i}) - R_f,$$

with $\mu^i(R^{\bar{\lambda}^i}) = \sum_k \bar{\lambda}_k^i \mu_k^i$. Hence, we have shown:

Proposition 2.2 (Individual Security Market Line) *For any investor i the risk premium of any asset k is proportional to the risk premium of his portfolio, where the factor of proportionality is given by the covariance of the return of asset k with investor i's portfolio divided by the variance of i's portfolio and risk premia are determined according to μ^i:*

$$\mu_k^i - R_f = \underbrace{\frac{\mathrm{COV}(R^k, R^{\bar{\lambda}^i})}{\sigma^2(R^{\bar{\lambda}^i})}}_{\beta^{i,k}}(\mu^i(R^{\bar{\lambda}^i}) - R_f),\ k = 1, \ldots, K. \tag{10}$$

We see that an investor i will hold the market portfolio if $\mu^i = \bar{\mu}$, i.e. if his expectations coincide with the average expectations in the market.

As a last point of this section we show that our model can address the phenomenon of underdiversification. There is considerable empirical evidence showing that the average investor is heavily underdiversified compared to the market portfolio (cf. Odean (1999), Goetzmann and Kumar (2005) and Polkovnichenko (2005)). Our results show that underdiversification is consistent with optimal investment in an economy, where investors have heterogenous beliefs. To see this, consider the following simple example:

Example 2.1 Let there be two investors $i = 1, 2$, and two risky assets $k = 1, 2$. Let the covariance matrix of asset returns be given by

$$\mathrm{COV} = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix},$$

where $\sigma_1^2 > 0$ and $\sigma_2^2 > 0$. Moreover, assume that investor i's belief about expected asset returns is given by

$$\mu^1 = \begin{pmatrix} d \\ R_f \end{pmatrix} \text{ and } \mu^2 = \begin{pmatrix} R_f \\ d \end{pmatrix},$$

where $d > R_f$. Then, it is straightforward to show that

$$\bar{\lambda}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \bar{\lambda}^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, investor 1 invests only into asset 1 and investor 2 only invests into asset 2, while the market portfolio is given by

$$\lambda^M = \begin{pmatrix} r^1 \\ r^2 \end{pmatrix}.$$

Thus, in equilibrium each investor is underdiversified compared to the market portfolio.

3 Active and Passive Investment

So far we have not said anything about what determines the belief of investor i. Suppose that each investor can choose to invest a fraction $K^i > 0$ of his initial wealth w_0^i in order to generate a belief μ^i about the average return of the assets.[6]

[6]Since investors take asset prices as given this is equivalent to assuming that investors can invest in generating a belief m^i about average asset payoffs.

If the investor does not invest in his own belief he observes the market belief without incurring any costs, i.e. the average market expectation $\bar{\mu}$ (which, of course, is endogenous). Let $\tilde{\mu}^i \in \{\mu^i, \bar{\mu}\}$ be investor i's belief. If $\tilde{\mu}^i = \mu^i$, we call i an *active investor* and if $\tilde{\mu}^i = \bar{\mu}$, then i is called a *passive investor*.

Recall that

$$\bar{\mu} = \sum_i a^i \tilde{\mu}^i, \tag{11}$$

where $a^i = \frac{w_0^i}{\gamma^i}\left(\sum_j \frac{w_0^j}{\gamma^j}\right)^{-1}$. As we have seen before, given belief $\tilde{\mu}^i$, investor i optimally chooses

$$\lambda^i(\tilde{\mu}^i) = \mathrm{COV}^{-1}\,\frac{\tilde{\mu}^i - R_f e}{\gamma^i},$$

and invests $1-\sum_{k=1}^K \lambda_k^i(\tilde{\mu}^i)$ into the riskless asset. Hence, he obtains the portfolio return

$$R(\tilde{\mu}^i) = R_f + \sum_{k=1}^K \lambda_k^i(\tilde{\mu}^i)(R^k - R_f).$$

Clearly, a passive investor will hold the market portfolio of risky assets.

We assume that investors ex post observe the true expected returns $\hat{\mu}$.[7] We denote by $U_{\hat{\mu}}^i(\tilde{\mu}^i)$ investor i's ex post utility under the true expected returns, i.e.

$$U_{\hat{\mu}}^i(\tilde{\mu}^i) = \mathbb{E}(R(\tilde{\mu}^i)) - \frac{\gamma^i}{2}\sigma^2(R(\tilde{\mu}^i)).$$

Hence,

$$\begin{aligned} U_{\hat{\mu}}^i(\tilde{\mu}^i) &= R_f + \frac{1}{\gamma^i}(\tilde{\mu}^i - R_f e)^T \mathrm{COV}^{-1}(\hat{\mu} - R_f e) \\ &\quad -\frac{\gamma^i}{2}\left(\frac{\tilde{\mu}^i - R_f e}{\gamma^i}\right)^T \mathrm{COV}^{-1}\left(\frac{\tilde{\mu}^i - R_f e}{\gamma^i}\right) \\ &= R_f + \frac{1}{\gamma^i}(\tilde{\mu}^i - R_f e)^T \mathrm{COV}^{-1}\left(\hat{\mu} - \frac{1}{2}\tilde{\mu}^i - \frac{1}{2}R_f e\right). \end{aligned}$$

Observe that $U_{\hat{\mu}}^i(\tilde{\mu}^i)$ is maximized for $\tilde{\mu}^i = \hat{\mu}$, i.e. for the case, where i has correct beliefs. Investor i chooses $\tilde{\mu}^i = \mu^i$ if

$$U_{\hat{\mu}}^i(\mu^i) - K^i \geq U_{\hat{\mu}}^i(\bar{\mu})$$

[7]The underlying idea is that investors do not revise their investment strategy frequently so that they get enough observations of the asset returns in order to get a very precise estimate of the true expected returns.

and $\tilde{\mu}^i = \bar{\mu}$ otherwise.[8]

We define the following scalar product on $\mathbb{R}^K$:

$$< x, y > := x^T \text{COV}^{-1} y, \quad x, y \in \mathbb{R}^K. \tag{12}$$

Observe that $< \cdot, \cdot >$ is indeed a scalar product. In particular, $< \cdot, \cdot >$ is positive definite since COV and hence COV^{-1} is positive definite. Using $< \cdot, \cdot >$ we define the following norm on $\mathbb{R}^K$:

$$\|x\| := \sqrt{< x, x >} = \sqrt{x^T \text{COV}^{-1} x}, \quad x \in \mathbb{R}^K. \tag{13}$$

With respect to this norm, $U^i_{\hat{\mu}}(\mu)$ is decreasing in the distance of μ to $\hat{\mu}$ (the true expectations) as is shown in the following lemma.

Lemma 3.1 *Let $\mu, \mu' \in \mathbb{R}^K$. Then*

$$U^i_{\hat{\mu}}(\mu) - U^i_{\hat{\mu}}(\mu') = \frac{1}{2\gamma^i} \left(\|\hat{\mu} - \mu'\|^2 - \|\hat{\mu} - \mu\|^2 \right).$$

Hence,

$$\|\hat{\mu} - \mu\| < \|\hat{\mu} - \mu'\| \iff U^i_{\hat{\mu}}(\mu) > U^i_{\hat{\mu}}(\mu').$$

Proof: Simple computation:

$$\begin{aligned} U^i_{\hat{\mu}}(\mu) - U^i_{\hat{\mu}}(\mu') &= \frac{1}{\gamma^i} \Big[< \mu - R_f e, \hat{\mu} - \frac{1}{2}\mu - \frac{1}{2} R_f e > \\ &\qquad - < \mu' - R_f e, \hat{\mu} - \frac{1}{2}\mu' - \frac{1}{2} R_f e > \Big] \\ &= \frac{1}{\gamma^i} \Big[< \mu, \hat{\mu} - \frac{1}{2}\mu > - < \mu', \hat{\mu} - \frac{1}{2}\mu' > \Big] \\ &= \frac{1}{2\gamma^i} \left(\|\hat{\mu} - \mu'\|^2 - \|\hat{\mu} - \mu\|^2 \right). \end{aligned}$$

□

From Lemma 3.1 it follows that investor i chooses $\tilde{\mu}^i = \mu^i$ if and only if

$$\|\bar{\mu} - \hat{\mu}\|^2 - \|\mu^i - \hat{\mu}\|^2 \geq 2K^i \gamma^i. \tag{14}$$

[8] Note that the costs K^i are measured in terms of wealth since the investor's utility is linear in expected wealth.

The decision to become active or remain passive thus depends on the accuracy of the average expectations $\|\bar{\mu}-\hat{\mu}\|$ as well as on the accuracy of the investor's belief $\|\mu^i - \hat{\mu}\|$. We say that $\|\bar{\mu} - \hat{\mu}\|$ measures the "efficiency of the market", while $\|\mu^i - \hat{\mu}\|$ measures the individual "skill" of investor i. Hence, ceteris paribus, investor i is more inclined to be passive the more risk averse he is, the lower his skill, the higher his investment cost and the more efficient the market is.

Observe that $\|\mu^i-\hat{\mu}\|$ and $\|\bar{\mu}-\hat{\mu}\|$ are independent of asset prices q. To see this recall that Δ is the diagonal matrix with $\Delta_{k,k} = 1/q^k$ for all k. Hence, $\mu^i = \Delta m^i$ for all i and $\bar{\mu} = \Delta\bar{m}$, where $\bar{m} = \sum_i a^i m^i$. If we define $\hat{m}$ by $\hat{m}_k := \mathbb{E}(A^k)$ for all k to be the vector of true expected asset payoffs, then

$$\begin{aligned}
\|\mu^i - \hat{\mu}\|^2 &= (\mu^i - \hat{\mu})^T \mathrm{COV}^{-1}(\mu^i - \hat{\mu}) \\
&= (m^i - \hat{m})^T \Delta\Delta^{-1} S^{-1} \Delta^{-1} \Delta (m^i - \hat{m}) \\
&= (m^i - \hat{m})^T S^{-1} (m^i - \hat{m}) \text{ for all } i,
\end{aligned}$$

which is independent of q. Similarly,

$$\|\bar{\mu} - \hat{\mu}\|^2 = (\bar{m} - \hat{m})^T S^{-1} (\bar{m} - \hat{m})$$

is independent of q. Since $\|\mu^i - \hat{\mu}\|$ is independent of asset prices, the skill of investor i is independent of the portfolios chosen by other investors. Hence, the skill of an investor is a truly personal characteristic.

We are now in the position to define a stability notion of CAPM-equilibria with information acquisition. We say that a profile with heterogenous beliefs is stable if every investor has deviated from the average belief if and only if this is in line with his optimal information acquisition choice. In particular, at a stable profile of beliefs no investor has an incentive to modify his belief by acquiring information.

Definition 3.1 *The profile $\tilde{\mu} = (\tilde{\mu}^1, \ldots, \tilde{\mu}^I)$ is* ***stable****, if the following condition is satisfied: For all i,*

$$\|\bar{\mu} - \hat{\mu}\|^2 - \|\mu^i - \hat{\mu}\|^2 \geq 2K^i\gamma^i \iff \tilde{\mu}^i = \mu^i. \tag{15}$$

Let $\tilde{\mu} = (\tilde{\mu}^1, \ldots, \tilde{\mu}^I)$ be some profile of beliefs. Then, from (11) it follows that

$$\bar{\mu} = \left(\sum_{i:\tilde{\mu}^i = \mu^i} a^i \right)^{-1} \sum_{i:\tilde{\mu}^i = \mu^i} a^i \mu^i,$$

whenever $\{i : \tilde{\mu}^i = \mu^i\} \neq \emptyset$, and $\bar{\mu}$ is undetermined otherwise.

Proposition 3.1 *There exists no stable profile $\tilde{\mu}$ where some investor acquires information, i.e. where $\tilde{\mu}^i = \mu^i$ for some i.*

Proof: Suppose by way of contradiction that $\tilde{\mu}$ is stable and that $\{i : \tilde{\mu}^i = \mu^i\} \neq \emptyset$. W.l.o.g. let $\{i : \tilde{\mu}^i = \mu^i\} = \{1, \ldots, J\}$. Then

$$\bar{\mu} = \frac{1}{\bar{a}_J} \sum_{j=1}^{J} a^j \mu^j,$$

where $\bar{a}_J := \sum_{j=1}^{J} a^j$. W.l.o.g. let $\|\mu^1 - \hat{\mu}\| \leq \|\mu^2 - \hat{\mu}\| \leq \ldots \leq \|\mu^J - \hat{\mu}\|$. Then

$$\begin{aligned} \|\bar{\mu} - \hat{\mu}\| &= \frac{1}{\bar{a}_J} \| \sum_{j=1}^{J} a^j (\mu^j - \hat{\mu}) \| \\ &\leq \frac{1}{\bar{a}_J} \sum_{j=1}^{J} a^j \|\mu^j - \hat{\mu}\| \\ &\leq \|\mu^J - \hat{\mu}\| \end{aligned}$$

Hence, (15) is violated for $i = J$ contradicting the fact that $\tilde{\mu}$ is stable.

□

Hence, we obtain the paradoxical result that there cannot be active investment in a stable market. Whether or not passive investment leads to a stable situation now depends on how $\bar{\mu}$ (which is undetermined if all investors are passive) relates to the true beliefs $\hat{\mu}$: If the market is very "efficient," i.e. $\|\bar{\mu} - \hat{\mu}\|$ is close to zero, then (15) is violated for all i, so that every investor being passive ($\tilde{\mu}^i = \bar{\mu}$ for all i) is stable. If, on the contrary, $\|\bar{\mu} - \hat{\mu}\|$ is large, so that there exists an investor i, for whom active investment is profitable, i.e. (14) is satisfied, then passive investment is not stable. In other words, the standard CAPM with homogenous beliefs $\bar{\mu}$ that are close to the true beliefs $\hat{\mu}$ according to the efficiency measure $\|\bar{\mu} - \hat{\mu}\|$, is the only stable outcome of our model.

Proposition 3.2 *The profile $\tilde{\mu} = (\tilde{\mu}^1, \ldots, \tilde{\mu}^I)$ is stable if and only if there exists $\bar{\mu}$ such that*

(i) $\tilde{\mu}^i = \bar{\mu}$, *and*

(ii) $\|\bar{\mu} - \hat{\mu}\|^2 < 2K^i\gamma^i + \|\mu^i - \hat{\mu}\|^2$,

for all i.

Now we are in a position to address the structure of performance fees that are in line with the information acquisition decision of the investors. We have seen that there cannot be active investment in the long run. In the short run, however, in particular if the true belief $\hat{\mu}$ changes, there is a potential for active investment if the market is inefficient, i.e. $\|\bar{\mu} - \hat{\mu}\|$ large and the skill is high, i.e. $\|\mu^i - \hat{\mu}\|$ is small. Suppose now that an investor cannot invest actively on his own but has to invest into a fund if he wants to be active. This fund sells a portfolio λ which, from the perspective of investor i, corresponds to the belief

$$\mu^i = R_f e + \gamma^i \mathrm{COV}\lambda,$$

which follows from (2). The question then is, how the fee of the fund should look like in order to induce the investor to invest into the fund.

From our previous analysis we obtain two conditions:

(1) In order to give the fund manager the right incentives, the performance fee should be increasing in the skill of the manager, i.e. decreasing in $\|\mu^i - \hat{\mu}\|$, since $U^i_{\hat{\mu}}(\mu^i)$ is decreasing in $\|\mu^i - \hat{\mu}\|$.

(2) In order for the investor to become active, the fee must be bounded above by a function that is decreasing in the risk aversion of the investor and in the efficiency of the market.

We get the following result:

Corollary 3.1 *Any performance-fee $K^i = K^i(\|\mu - \hat{\mu}\|, \|\bar{\mu} - \hat{\mu}\|)$, that is decreasing in $\|\mu - \hat{\mu}\|$ and that satisfies*

$$K^i \leq \frac{1}{2\gamma^i}\left(\|\bar{\mu} - \hat{\mu}\|^2 - \|\mu - \hat{\mu}\|^2\right),$$

fulfills these conditions.

Hence, the performance fee should reward the skill of the manager but should also discourage the manager to hunt for investment opportunities in efficient markets. Moreover, comparing agents with different degrees of risk aversion, we find that the more risk averse agents have a lower willingness to pay for active portfolio management and therefore are more inclined to be passive.

Before we close this section we address the issue of erosion of investment opportunities. We have seen that only passive investment is stable. Nevertheless, in the short run, for example, due to changes in the exogenous uncertainty, some investors may find it profitable to become active. We will now show that active investment is profitable only if the investor's wealth is small relative to the aggregate wealth in the economy. In other words, profitable investment opportunities resulting from inefficient markets (i.e. $\|\bar{\mu} - \hat{\mu}\|$ large) erode if the investor accumulates too much wealth.

Proposition 3.3 *Let* $\left((w_0^{i,n})_i\right)_n$ *be a sequence of wealth profiles such that*

$$\lim_{n\to\infty} \frac{w_0^{i,n}}{\sum_j w_0^{j,n}} = 1$$

for some i. *Then*

$$\lim_{n\to\infty} \|\bar{\mu}^n - \hat{\mu}\| = \|\mu^i - \hat{\mu}\|,$$

where $\bar{\mu}^n = \sum_j a^{j,n}\tilde{\mu}^{j,n}$ *with* $\tilde{\mu}^{i,n} = \mu^i, \tilde{\mu}^{j,n} \in \{\mu^j, \bar{\mu}^n\}$ *for all* $j \neq i$, *and* $a^{j,n} = \frac{w_0^{j,n}}{\gamma^j}\left(\sum_h \frac{w_0^{h,n}}{\gamma^h}\right)^{-1}$ *for all* j *and all* n.

Proof: From $w_0^{i,n}/\left(\sum_j w_0^{j,n}\right) \to 1$ it follows that $w_0^{j,n}/w_0^{i,n} \to 0$ for all $j \neq i$. This implies

$$a^{i,n} = \frac{1}{\gamma^i}\left(\sum_j \frac{w_0^{j,n}}{w_0^{i,n}\gamma^j}\right)^{-1} \to 1.$$

Hence, $\bar{\mu}^n = \left(\sum_{j:\tilde{\mu}^{j,n}=\mu^j} a^{j,n}\right)^{-1} \sum_{j:\tilde{\mu}^{j,n}=\mu^j} a^{j,n}\mu^j \to \mu^i$ which implies that $\|\bar{\mu}^n - \hat{\mu}\| \to \|\mu^i - \hat{\mu}\|$.

□

4 The Alpha

The "alpha" is one of the most used buzzwords in the finance jargon. It measures the deviation of mean asset returns from the security market line. Investment funds, in particular hedge funds, claim to generate a positive alpha in order to attract assets under management. However, so far there is no theory that explains the existence of a nonzero alpha. A nonzero alpha is in contradiction to the standard CAPM with homogenous beliefs. Under homogenous and correct beliefs, everyone holds the market portfolio and hence, each asset generates an alpha of zero. To see this define the (ex post) alpha of asset k by

$$\hat{\alpha}_k := \hat{\mu}_k - R_f - \beta^{M,k}(\hat{\mu}^M - R_f),$$

where $\hat{\mu}^M := \sum_k \lambda_k^M \hat{\mu}_k$ is the true return of the market portfolio. If $\mu^i = \hat{\mu}$ for all i, then $\hat{\alpha}_k = 0$ follows from Proposition 2.2. Hence, in the standard CAPM there is no portfolio which generates a positive alpha. By contrast, under heterogenous beliefs, there typically exist portfolios generating a positive alpha. To see this recall that in equilibrium

$$\bar{\alpha}_k := \bar{\mu}_k - R_f - \beta^{M,k}(\bar{\mu}^M - R_f) = 0$$

for all k by Proposition 2.1. Hence, if average expectations differ from true expectations, $\bar{\mu} \neq \hat{\mu}$, typically there will exist k such that $\hat{\alpha}_k \neq \bar{\alpha}_k = 0$ and hence there will exist a portfolio of risky assets $\bar{\lambda}$, which generates a positive alpha, i.e. $\sum_k \bar{\lambda}_k \hat{\alpha}_k > 0$.

Thus, our CAPM model with heterogenous expectations can explain the existence of a nonzero alpha in equilibrium. However, it will turn out that the hunt for alpha opportunities is a zero sum game and that alpha opportunities erode whenever the investor accumulates too much wealth in the economy. Moreover, we will argue that a positive alpha is not necessarily a good criterion for active portfolio management. Hence, our model on the one hand provides a thorough foundation for the alpha and on the other hand casts serious doubt on its use in practical financial decisions.

In order to derive these results we define the ex post or true alpha of investor

i's portfolio as

$$\hat{\alpha}^i := \sum_k \bar{\lambda}^i_k \hat{\alpha}_k,$$

and obtain

$$\begin{aligned}
\sum_i w^i_f \hat{\alpha}^i &= \sum_i r^i \left(\sum_j w^j_f \right) \sum_k \bar{\lambda}^i_k \hat{\alpha}_k \\
&= \left(\sum_j w^j_f \right) \sum_k \hat{\alpha}_k \lambda^M_k \\
&= \left(\sum_j w^j_f \right) \left(\hat{\mu}^M - R_f \sum_k \lambda^M_k - (\hat{\mu}^M - R_f) \sum_k (\beta^{M,k} \lambda^M_k) \right) \\
&= 0.
\end{aligned}$$

Hence, since $w^j_f > 0$ for all j, an investor i can generate a positive alpha if and only if there is another investor j who generates a negative alpha. We state this zero sum game property in the following proposition.

Proposition 4.1 (The Hunt for Alpha Opportunities is a Zero Sum Game) *In equilibrium*

$$\sum_i w^i_f \hat{\alpha}^i = 0.$$

Let $\left((w^{i,n}_0)_i \right)_n$ be a sequence of wealth profiles. Then, by $R^{M,n}$ we denote the equilibrium return of the market portfolio under the wealth profile $(w^{i,n}_0)_i$ and we let $\hat{\mu}^{M,n}$ denote the expectation of $R^{M,n}$ under the true beliefs. By $R^{k,n}$ we denote the equilibrium return of asset k under the wealth profile $(w^{i,n}_0)_i$ and we let $\hat{\mu}^n_k$ denote the expectation of $R^{k,n}$ under the true beliefs. Finally, for all k, we define

$$\hat{\alpha}^n_k = \hat{\mu}^n_k - R_f - \beta^{M,k}_n (\hat{\mu}^{M,n} - R_f),$$

where $\beta^{M,k}_n = \mathrm{COV}(R^{k,n}, R^{M,n}) / \sigma^2(R^{M,n})$.

Proposition 4.2 (Erosion of Alpha Opportunities) *Let* $\left((w_0^{i,n})_i\right)_n$ *be a sequence of wealth profiles such that* $\lim_{n\to\infty} w_0^{i,n} = w_0^i$ *and*

$$\lim_{n\to\infty} \frac{w_0^{i,n}}{\sum_j w_0^{j,n}} = 1$$

for some i. *Then*

$$\lim_{n\to\infty} \hat{\alpha}^{i,n} = 0,$$

where

$$\hat{\alpha}^{i,n} = \sum_k \bar{\lambda}_k^{i,n} \hat{\alpha}_k^n \quad \textit{for all } n.$$

Proof: Let q^n be the equilibrium price vector for the wealth profile $(w_0^{i,n})_i$. Then, from $\lim_{n\to\infty} w_0^{i,n} / \left(\sum_j w_0^{j,n}\right) \to 1$ and (3) it follows that

$$\lim_{n\to\infty} q^n = \frac{1}{R_f \frac{w_0^i}{\gamma^i}} \left(\frac{w_0^i}{\gamma^i} m^i - Se \right) =: q.$$

Let $\lambda^{j,n}$ be the equilibrium portfolio of investor j at the wealth profile $(w^{\iota,n})_\iota$ and define

$$r^{j,n} = \frac{(1-\lambda_0^{j,n}) w_0^{j,n}}{\sum_\iota (1-\lambda_0^{\iota,n}) w_0^{\iota,n}}$$

for all j and n. Since $q^n \to q$, for all j, there exists λ_0^j such that $\lim_{n\to\infty} \lambda_0^{j,n} = \lambda_0^j$. Hence, from $\lim_{n\to\infty} w_0^{i,n} / \left(\sum_j w_0^{j,n}\right) \to 1$ it follows that

$$\lim_{n\to\infty} r^{i,n} = \frac{(1-\lambda_0^i) w_0^{i,n}}{\sum_j (1-\lambda_0^j) w_0^{j,n}} = 1,$$

which implies that

$$\lim_{n\to\infty} \lambda^{M,n} = \lim_{n\to\infty} \sum_j r^{j,n} \bar{\lambda}^{j,n} = \bar{\lambda}^i.$$

Since $q = \lim_{n\to\infty} q^n$ it follows that

$$\lim_{n\to\infty} R^{k,n} = \frac{A^k}{q^k} =: R^k \text{ for all } k,$$

and

$$\lim_{n\to\infty} \hat{\mu}_k^n = \frac{\mathbb{E}(A^k)}{q^k} =: \hat{\mu}^k \text{ for all } k.$$

Moreover,

$$\lim_{n\to\infty} R^{M,n} = \lim_{n\to\infty} \sum_k \lambda_k^{M,n} R^{k,n} = R\bar{\lambda}^i.$$

Hence,

$$\begin{aligned}\lim_{n\to\infty} \hat{\alpha}_k^n &= \lim_{n\to\infty} \left[\hat{\mu}_k^n - R_f - \frac{\mathrm{COV}(R^{k,n}, R^{M,n})}{\sigma^2(R^{M,n})}(\hat{\mu}^{M,n} - R_f)\right] \\ &= \hat{\mu}_k - R_f - \frac{\mathrm{COV}(R^k, R\bar{\lambda}^i)}{\sigma^2(R\bar{\lambda}^i)}(\hat{\mu}(R\bar{\lambda}^i) - R_f),\end{aligned}$$

where $\hat{\mu}(R\bar{\lambda}^i) = \sum_k \bar{\lambda}_k^i \hat{\mu}_k$. This implies

$$\lim_{n\to\infty} \hat{\alpha}^{i,n} = \lim_{n\to\infty} \sum_k \bar{\lambda}_k^{i,n} \hat{\alpha}_k^n = 0$$

as claimed.

□

The following examples show that there can be a positive alpha in equilibrium but that the sign of alpha gives no appropriate decision criterion for active investment. The first example shows that $\hat{\alpha}^i$ can be positive although $\|\bar{\mu}-\hat{\mu}\| < \|\mu^i-\hat{\mu}\|$ so that investor i prefers to be passive at the given belief profile.

Example 4.1 Let $R_f = 1$ and let there be two risky assets and three investors $i = 1, 2, 3$, having the following characteristics:

$$\mu^1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix},\ \mu^2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mu^3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$\gamma^1 = \gamma^2 = \gamma^3 = 2$$
$$w_0^1 = w_0^2 = w_0^3 = 5$$

COV is given by

$$\mathrm{COV} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

and the true beliefs are

$$\hat{\mu} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Suppose now that all investors are active. We have $a^1 = a^2 = a^3 = 1/3$ and hence

$$\bar{\mu} = a^1\mu^1 + a^2\mu^2 + a^3\mu^3 = \begin{pmatrix} 2 \\ \frac{4}{3} \end{pmatrix}.$$

We obtain

$$\begin{aligned} \|\hat{\mu} - \mu^1\|^2 &= \tfrac{1}{2}, \\ \|\hat{\mu} - \mu^2\|^2 &= 0, \\ \|\hat{\mu} - \mu^3\|^2 &= 1, \\ \|\hat{\mu} - \bar{\mu}\|^2 &= \tfrac{1}{16}. \end{aligned}$$

Hence, investors 1 and 3 prefer to be passive for all costs K^1, respectively, K^3. Nevertheless, investor 1 generates a positive alpha when he is active:

The optimal portfolios of the investors (everyone is active!) are

$$\lambda^1 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \ \lambda^2 = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}, \ \lambda^3 = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}.$$

Hence,

$$\bar{\lambda}^1 = \bar{\lambda}^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \bar{\lambda}^3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the market portfolio is

$$\lambda^M = \sum_i r^i \bar{\lambda}^i = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix},$$

and hence

$$\beta^M = \frac{\mathrm{COV}\lambda^M}{(\lambda^M)^T \mathrm{COV}\lambda^M} = \begin{pmatrix} \frac{6}{5} \\ \frac{2}{5} \end{pmatrix}.$$

This implies

$$\hat{\alpha} = \hat{\mu} - R_f e - \beta^M(\hat{\mu}^M - R_f) = \begin{pmatrix} \frac{1}{10} \\ -\frac{3}{10} \end{pmatrix},$$

from which we compute

$$\begin{aligned} \hat{\alpha}^1 &= \hat{\alpha}^T \bar{\lambda}^1 = 1/10, \\ \hat{\alpha}^2 &= \hat{\alpha}^T \bar{\lambda}^2 = 1/10, \\ \hat{\alpha}^3 &= \hat{\alpha}^T \bar{\lambda}^3 = -3/10. \end{aligned}$$

Hence, we see that investor 1 generates a positive alpha by being active although he prefers to be passive.

The second example shows that $\hat{\alpha}^i$ can be negative although $\|\bar{\mu} - \hat{\mu}\| > \|\mu^i - \hat{\mu}\|$.

Example 4.2 Let the asset market structure be the same as in the previous example. Let there be four investors $i = 1, 2, 3, 4$, having the following characteristics:

$$\mu^1 = \begin{pmatrix} 6 \\ 1 \end{pmatrix}, \ \mu^2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mu^3 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \ \mu^4 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$
$$\gamma^1 = \gamma^2 = \gamma^3 = \gamma^4 = 2$$
$$w_0^1 = w_0^2 = w_0^3 = w_0^4 = 10$$

The true beliefs are

$$\hat{\mu} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Suppose now that all investors are active. We have $a^1 = a^2 = a^3 = a^4 = 1/4$ and hence

$$\bar{\mu} = a^1\mu^1 + a^2\mu^2 + a^3\mu^3 + a^4\mu^4 = \begin{pmatrix} 3 \\ \frac{11}{4} \end{pmatrix}.$$

We obtain

$$\begin{aligned} \|\hat{\mu} - \mu^1\|^2 &= \tfrac{17}{2}, \\ \|\hat{\mu} - \mu^2\|^2 &= \tfrac{1}{2}, \\ \|\hat{\mu} - \mu^3\|^2 &= \tfrac{1}{2}, \\ \|\hat{\mu} - \mu^4\|^2 &= 5, \\ \|\hat{\mu} - \bar{\mu}\|^2 &= \tfrac{25}{32}. \end{aligned}$$

Hence, investors 2 and 3 prefer to be active for all costs K^2, respectively, K^3. Nevertheless, investor 2 generates a negative alpha when he is active:

The optimal portfolios of the investors (everyone is active!) are

$$\lambda^1 = \begin{pmatrix} \frac{5}{4} \\ 0 \end{pmatrix}, \ \lambda^2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}, \ \lambda^3 = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}, \ \lambda^4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence,

$$\bar{\lambda}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \bar{\lambda}^2 = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix},\ \bar{\lambda}^3 = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix},\ \bar{\lambda}^4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the market portfolio is

$$\lambda^M = \sum_i r^i \bar{\lambda}^i = \begin{pmatrix} \frac{8}{15} \\ \frac{7}{15} \end{pmatrix},$$

and hence

$$\beta^M = \frac{\mathrm{COV}\lambda^M}{(\lambda^M)^T \mathrm{COV}\lambda^M} = \begin{pmatrix} \frac{110}{113} \\ \frac{105}{113} \end{pmatrix}.$$

This implies

$$\hat{\alpha} = \hat{\mu} - R_f e - \beta^M(\hat{\mu}^M - R_f) = \begin{pmatrix} -\frac{7}{113} \\ \frac{8}{113} \end{pmatrix},$$

from which we compute

$$\begin{aligned} \hat{\alpha}^1 &= \hat{\alpha}^T \bar{\lambda}^1 = -\tfrac{7}{113}, \\ \hat{\alpha}^2 &= \hat{\alpha}^T \bar{\lambda}^2 = -\tfrac{2}{113}, \\ \hat{\alpha}^3 &= \hat{\alpha}^T \bar{\lambda}^3 = \tfrac{3}{113}, \\ \hat{\alpha}^4 &= \hat{\alpha}^T \bar{\lambda}^4 = \tfrac{8}{113} \end{aligned}$$

Hence, we see that investor 2 generates a negative alpha by being active although he prefers to be active if his costs K^i are sufficiently small.

References

AGARWAL, V., N. D. DANIEL, AND N. Y. NAIK (2005): "Role of Managerial Incentives, Flexibility and Ability: Evidence from Performance and Money Flows in Hedge Funds," mimeo, Georgia State University and London Business School.

BERK, J. B., AND R. C. GREEN (2004): "Mutual Fund Flows and Performance in Rational Markets," *Journal of Political Economy*, 112, 1269–1295.

BROCK, W. A., AND C. H. HOMMES (1998): "Heterogeneous Beliefs and Routes to Chaos in a Simple Asset Pricing Model," *Journal of Economic Dynamics and Control*, 22, 1235–1274.

CHOPRA, V. K., AND W. T. ZIEMBA (1993): "The Effect of Errors in Mean, Variance and Co-Variance Estimates on Optimal Portfolio Choice," *Journal of Portfolio Management*, 19, 6–11.

GETMANSKY, M. (2004): "The Life Cycle of Hedge Funds: Fund Flows, Size and Performance," mimeo, MIT Sloan School of Management.

GOETZMANN, W. N., AND A. KUMAR (2005): "Why Do Individual Investors Hold Under-Diversified Portfolios?," Yale School of Management Working Papers ysm454, Yale School of Management, available at http://ideas.repec.org/p/ysm/somwrk/ysm454.html.

GRINOLD, R. C., AND R. N. KAHN (2000): *Active Porfolio Management: A Quantitative Approach for Providing Superior Returns and Controlling Risk.* McGraw-Hill, New York.

GROSSMAN, S. J., AND J. E. STIGLITZ (1980): "On the Impossibility of Informationally Efficient Markets," *American Economic Review*, 70, 393–408.

LINTNER, J. (1969): "The Aggregation of Investor's Diverse Judgments and Preferences in Purely Competitive Security Markets," *Journal of Financial and Quantitative Analysis*, 4, 347–400.

Litterman, R. (2003): "Why an Equilibrium Approach?," in *Modern Investment Management: An Equilibrium Approach*, ed. by B. Litterman, and the Quantitative Research Group Goldman Sachs Asset Management, chap. 1, pp. 3–6. Wiley, Hoboken, New Jersey.

Odean, T. (1999): "Do Investors Trade Too Much," *The American Economic Review*, 89, 1279–1298.

Polkovnichenko, V. (2005): "Household Portfolio Diversification: A Case for Rank Dependent Preferences," *The Review of Financial Studies*, 18, 1467–1502.